The Variable

Presented by the Saskatchewan Mathematics Teachers' Society

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Saskatchewan Mathematics Teachers' Society

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Cover art

"This photo was taken for the Math Photo 16 Challenge; the theme of the week was 'Symmetry.' To see all of the photos submitted to the challenge, head to https://mathphoto16.wordpress.com/ or search the #mathphoto16 hashtag on Twitter.

By the way, the spiraling pentagons featured in the photo were made by Christopher Danielson of Talking Math With Your Kids, and are available for order, along with other tiling shapes, mathematical toys, and books, on his website. Speaking from experience, they are sure to provide hours of fun and exploration for mathematicians young and old!"

Ilona Vashchyshyn

SMTS Executive (2016-2018)

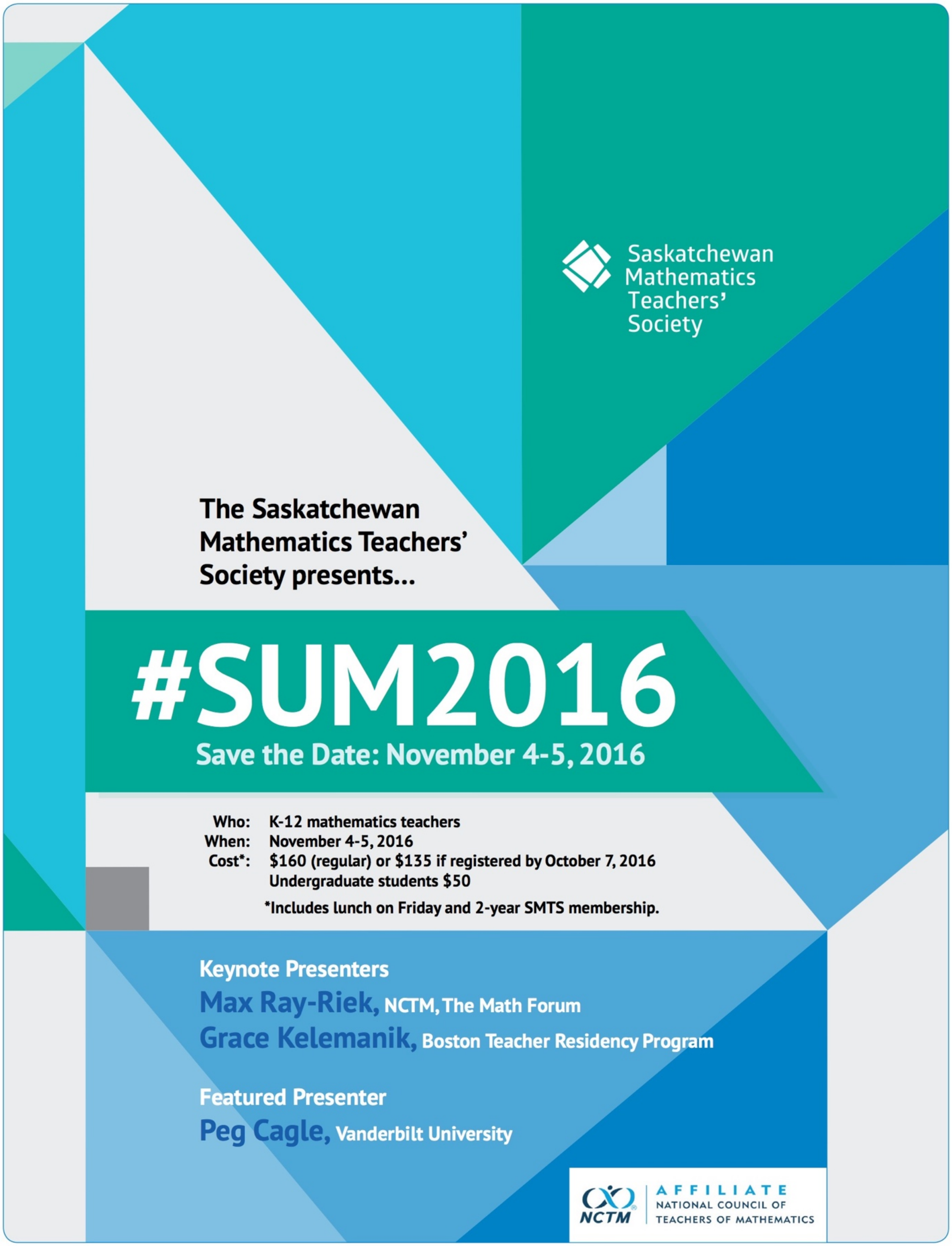
Notice to Readers

The views expressed in *The Variable* are those of the author(s), and not necessarily those of the Saskatchewan Mathematics Teachers' Society. In the publishing of any works herein, the editorial board has sought confirmation from all authors that personal quotes and examples have been included with the permission of those to whom they are attributable.

Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants, teacher educators, researchers, and students of all ages, although we encourage Saskatchewan teachers of mathematics as our main contributors. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Articles should be limited to 3000 words or less; authors should also include a photo and a short biographical statement of 75 words or less. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.


Saskatchewan Mathematics Teachers' Society
The Saskatchewan Mathematics Teachers' Society presents...
#SUM2016
Save the Date: November 4-5, 2016
Who: K-12 mathematics teachers
When: November 4-5, 2016
Cost*: $160 (regular) or $135 if registered by October 7, 2016
Undergraduate students $50
*Includes lunch on Friday and 2-year SMTS membership.
Keynote Presenters
Max Ray-Riek, NCTM, The Math Forum
Grace Kelemanik, Boston Teacher Residency Program
Featured Presenter
Peg Cagle, Vanderbilt University
NCTM
AFFILIATE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Message from the President

Welcome back! I hope your summer was exactly what you needed in order for this back-to-school season to feel exciting and full of promise. While I'm sad to see summer end—what is up with these suddenly rather chilly mornings and evenings?!—I love getting ready for a new year. September is my true "new year" and is far more likely to be full of "resolutions" than January.

The 2015-2016 school year was all about change and growth for me personally, professionally, and for the Saskatchewan Mathematics Teachers' Society. It was exciting and full of big plans. The SMTS held its first November Saskatchewan Understands Math (SUM) Conference, launched *The Variable*, and began publishing weekly blog posts. We also spent time making a three-year strategic plan and reviewing the services we provide to members. We hope that we're moving and changing in a direction that provides all Saskatchewan teachers (and beyond!) with better supports for teaching mathematics.

When I think about what the 2016-2017 school year will bring, again I see a strong parallel between my professional work and the work of the SMTS, because in both cases, it revolves around building community. We hope you'll join us again in November at the SUM Conference, and we look forward to growing our partnerships with the University of Regina, the University of Saskatchewan, and other SUM partners. We also hope you'll join us on Twitter: Our official handle is @SMTSca, and many of the SMTS directors can be found chatting about math there, too. It's so easy to feel isolated as a teacher, whether it's because you are literally the only math teacher for miles, or because you can't get out of your classroom to see any other math teachers! Luckily, you can join us for #skmathphoto or #skmathchat anytime.

There are many more exciting things in the works this year, but I need to save myself a few things to write about next month! So here's wishing you a wonderful back-to-school and a year full of learning and playing with your students and colleagues.

Michelle Naidu

Problems to Ponder

Welcome to this month's edition of Problems to Ponder! Pose them in your classroom as a challenge, or try them out yourself. Have an interesting solution? Send it to thevariable@smts.ca *for publication in a future issue of* The Variable!

Climbing snail [1]

A snail is at the bottom of a 30 foot well. The snail climbs up 4 feet each day and slides down 3 feet each night. How many days does it take for the snail to get out of the well?

What if the snail climbs up 4 feet each day and slides down 2 feet each night? What if it climbs up k feet each day and slides down n feet each night?

Three ants [2]

Three ants are sitting at the three corners of an equilateral triangle. Each ant randomly picks a direction and starts to move along the edge of the triangle. What is the probability that none of the ants collide?

Extend to other shapes (e.g., square? octagon? n-sided polygon?).

Sweet and sour [3]

Suppose that an ant will always position itself so that it's precisely twice as far from vinegar as it is from honey. If we put a dab of vinegar at a point A and a dab of honey at a point B and we release a troop of ants, what formation will they take up?

Sources

[1]*Adapted from* Coldwell, N. (n.d.). *A collection of quant riddles with answers.* Retrieved from http://puzzles.nigelcoldwell.co.uk/

[2]*Adapted from* Coldwell, N. (n.d.). *A collection of quant riddles with answers.* Retrieved from http://puzzles.nigelcoldwell.co.uk/

[3]*Adapted from* Barbeau, E. (1995). *After math*. Toronto, ON: Wall & Emerson.

Problems to Ponder, July Edition: Solutions

Edward Doolittle

The theme of this month's solutions is "getting your hands dirty." By that, I mean strategies like tabulating values, making long lists, counting, and doing lots of arithmetic. Because of time limits in testing and contests, we can become conditioned to look for relatively quick solutions to any problem, but problems that aren't specially designed for tests or contests can take a considerable amount of time and effort. Even for test and contest problems, getting your hands dirty can provide insights or practice with calculation skills, or just another valuable perspective on a problem.

The sixth cent [1]

You toss a fair coin 6 times, and I toss a fair coin 5 times. What is the probability that you get more heads than I do?

To reduce the chance of confusion, let's refer to the person who tosses the coin 6 times as Alice and the person who tosses the coin five times as Bob. First, we need to calculate the probabilities of various outcomes for Bob. The probability of tossing a coin five times and getting no heads is 1/2 x 1/2 x 1/2 x 1/2 x 1/2 = 1/32. There are five different ways of getting one head, so the probability of that outcome is 5/32. For two heads, you can count up the number of ways or just recognize that you are dealing with binomial coefficients, so the probability of Bob obtaining exactly three heads is 10/32. Notice that I'm not putting the fraction into lowest terms; lowest terms can actually be a hindrance in the middle of a large calculation. The remaining probabilities for Bob are 10/32 for 3 heads; 5/32 for 4 heads; and 1/32 for 5 heads.

For Alice, the denominators of the probabilities are $2^6 = 64$ and the numerators are binomial coefficients, which can be obtained from Bob's numerators by using Pascal's triangle, or whatever method you like. The probabilities are: 0 heads, 1/64; 1 head, 6/64; 2 heads, 15/64; 3 heads, 20/64; 4 heads, 15/64; 5 heads, 6/64; 6 heads: 1/64.

Now we need to figure out the joint probabilities of two simultaneous outcomes, one for Alice and one for Bob. For example, the probability of Alice obtaining 4 heads while Bob obtains 2 is given by the product 15/64 x 10/32 = 150/2048, since the two events are independent. The probabilities of all other pairs of possibilities can be calculated in a similar manner. Thus, we can make a table of all of the possibilities. (Since the denominators in each case are the same, namely 2048, I have left the denominators out of the table, which is one benefit of not putting fractions in lowest terms.) The complete set of joint probabilities is as follows:

[1] *Adapted from* Barbeau, E. J., Klamkin, M. S., & Moser, W. O. J. (1995). *Five hundred mathematical challenges.* USA: The Mathematical Association of America.

	A0	A1	A2	A3	A4	A5	A6
B0	1x1 = 1	1x6 = 6	1x15 = 15	1x20 = 20	1x15 = 15	1x6 = 6	1x1 = 1
B1	5x1 = 5	5x6 = 30	5x15 = 75	5x20 = 100	5x15 = 75	5x6 = 30	5x1 = 5
B2	10x1 = 10	10x6 = 60	10x15 = 150	10x20 = 200	10x15 = 150	10x6 = 60	10x1 = 10
B3	10x1 = 10	10x6 = 60	10x15 = 150	10x20 = 200	10x15 = 150	10x6 = 60	10x1 = 10
B4	5x1 = 5	5x6 = 30	5x15 = 75	5x20 = 100	5x15 = 75	5x6 = 30	5x1 = 5
B5	1x1 = 1	1x6 = 6	1x15 = 15	1x20 = 20	1x15 = 15	1x6 = 6	1x1 = 1

I have highlighted the cells in the table with outcomes for which Alice has more heads than Bob. Adding up all of the probabilities, we have (6 + 15 + 75 + 20 + 100 + 200 + 15 + 75 + 150 + 150 + 6 + 30 + 60 + 60 + 30 + 1 + 5 + 10 + 10 + 5 + 1)/2048 = 1024/2048 = 1/2.

Now, not only do we have an answer to the original question, we should also have some insight into whether – and how – you might be able to find the answer more quickly. The answer, 1/2, is suspiciously simple, so we should spend some time thinking about whether there is a slicker way to get it. Notice that we can rotate the shaded part by a half-turn to exactly cover the shaded part. Since the shaded part equals the unshaded part, and the table has total probability 1, the probabilities of both the shaded part and the unshaded part must be 1/2.

With that insight, you might be able to see how to construct an argument that doesn't require all the arithmetic we used. Hint: A half-turn is the same as a flip in the vertical axis followed by a flip in the horizontal axis. What does a flip in the vertical axis represent? What does a flip in the horizontal axis represent?

Dueling dice [2]

Consider the following four dice and the numbers on their faces:

- *Red : 0, 1, 7, 8, 8, 9*
- *Blue: 5, 5, 6, 6, 7, 7*
- *Green: 1, 2, 3, 9, 10, 11*
- *Black: 3, 4, 4, 5, 11, 12*

The dice are used to play the following game for two people. Player 1 chooses a die, then player 2 chooses a die. Then, each player rolls their die. The player with the highest number showing gets a point. The first player to get 7 points wins the game. If you are Player 1, which die should you choose? If you are Player 2, which die should you choose?

For this problem, we construct six tables, one for each pair of dice, comparing outcomes. The red, blue, and green dice are labelled R, B, G, respectively, while the black die is labelled K to avoid confusion with B.

[2] *Adapted from Duelling dice.* (n.d.). Retrieved from Mathematics Centre website: http://mathematicscentre.com/taskcentre/046dueld.htm

	0	1	7	8	8	9
5	B	B	R	R	R	R
5	B	B	R	R	R	R
6	B	B	R	R	R	R
6	B	B	R	R	R	R
7	B	B		R	R	R
7	B	B		R	R	R

	0	1	7	8	8	9
3	K	K	R	R	R	R
4	K	K	R	R	R	R
4	K	K	R	R	R	R
5	K	K	R	R	R	R
11	K	K	K	K	K	K
12	K	K	K	K	K	K

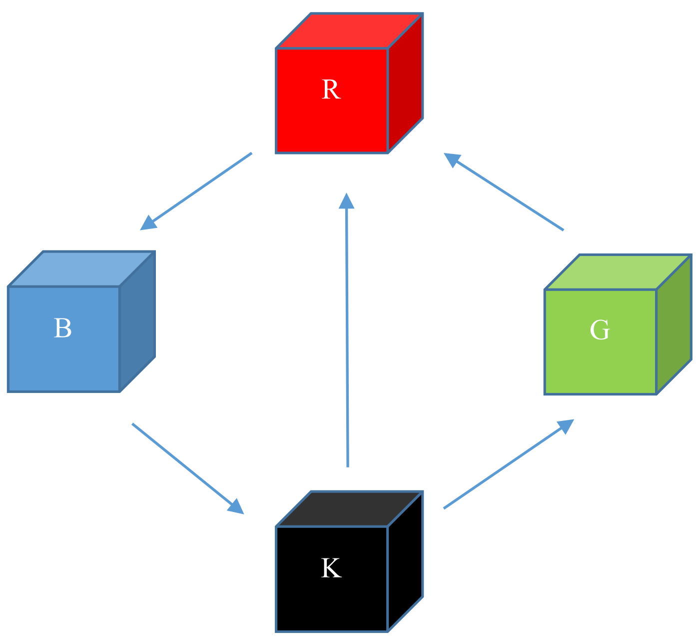
	5	5	6	6	7	7
3	B	B	B	B	B	B
4	B	B	B	B	B	B
4	B	B	B	B	B	B
5			B	B	B	B
11	K	K	K	K	K	K
12	K	K	K	K	K	K

	0	1	7	8	8	9
1	G		R	R	R	R
2	G	G	R	R	R	R
3	G	G	R	R	R	R
9	G	G	G	G	G	
10	G	G	G	G	G	G
11	G	G	G	G	G	G

	5	5	6	6	7	7
1	B	B	B	B	B	B
2	B	B	B	B	B	B
3	B	B	B	B	B	B
9	G	G	G	G	G	G
10	G	G	G	G	G	G
11	G	G	G	G	G	G

	1	2	3	9	10	11
3	K	K		G	G	G
4	K	K	K	G	G	G
4	K	K	K	G	G	G
5	K	K	K	G	G	G
11	K	K	K	K	K	
12	K	K	K	K	K	K

We can see from the tables that R "dominates" B—that is, in a single throw (or seven throws), the expected number of times that R beats B is greater than 1/2. Similarly, K dominates R, B dominates K, G dominates R, B ties with G, and K dominates G. In order to keep all of this data straight, we can use a picture called a directed graph or digraph. In the digraph, an arrow from A to B means that A dominates B.



An interesting phenomenon should be apparent from the above graph. Compare the situation of the dice with inequalities of numbers. If A, B, C are numbers, A < B, and B < C, then we can conclude that A < C. Inequality is transitive. On the other hand, our dice are "nontransitive," meaning that if A is better than B and B is better than C, we can't conclude that A is better than C. The presence of cycles in the directed graph (e.g., R → B → K → R or R → B → K → G → R) is a visual illustration that the dice are nontransitive.

Player 2 is expected to win this game. No matter which die Player 1 selects, Player 2 can move backwards along an arrow to find a die that dominates Player 1's choice. However, since the game depends to some degree on chance, Player 1 can minimize her odds of losing. One thing she might do is hope that Player 2 hasn't made such a careful analysis of the problem, and might carelessly (randomly, say) pick his die. In that case, Player 1's best choice is the black K die, because it dominates two other dice (G and R), giving a 2/3 chance of having the dominant die if Player 2 chooses randomly. If Player 2 does choose the blue die, however, Player 1 has a very high chance of losing in the rolling phase because B dominates K, with probability 22/36 of winning a single roll.

However, in game theory, we assume that Player 2 will play optimally, always choosing the worst case scenario for Player 1. Given that assumption, Player 1's strategy will be to minimize the worst case scenario. Let's consider each of the dice in turn. We've already seen the worst case scenario if Player 1 chooses K. If Player 1 chooses B, the worst case scenario occurs when Player 2 chooses R, in which case Player 2 will again win 22/36 of the rolls, on average. If Player 1 chooses R, Player 2 can choose G, winning 22/36 of the rolls on average, or Player 2 can choose K, winning 20/36 of the rolls; therefore, Player 2 should choose G. Finally, if Player 1 chooses G, Player 2 should choose K, again winning 22/36 of the rolls.[3]

It looks like no matter what choice Player 1 makes, the worst case scenario is the same. Player 1 can't do better than losing 22/36 of the rolls against a well-informed opponent. Her only hope is to choose the black die and hope player 2 hasn't analyzed the game as thoroughly as we have, which isn't really a strategy in the game theory sense.

There is no straightforward shortcut to solving this problem that I can see. There may be a theory of nontransitive dice that could help, but I'm not aware of any such theory. On the other hand, as a first step toward gaining a better understanding of nontransitive dice, you should try constructing your own set of three nontransitive dice. Ambitiously, you can try to use all the numbers from 1 to 9. Since there are 18 faces on the three dice, let's stipulate that each number appears twice on one die (so if the number 3, say, appears on a die, then it appears exactly twice on that die, and not on any other die). Each die will have just three distinct numbers. You may find an item in last month's solutions to be helpful.

[3] The calculations show that these dice are a variation of Efron's dice, a set of four nontransitive dice such that the probabilities of A winning against B, B against C, C against D, and D against A are all the same. See http://mathworld.wolfram.com/EfronsDice.html

Two too many dice [4]

Suppose you have a clear, sealed cube containing three smaller, indistinguishable six-sided dice. How can you use this three-in-one die to simulate a single, six-sided die? (Bonus: How can you use the three-in-one die to simulate two *six-sided dice?)*

First, we need to understand what is meant by indistinguishable. In order to explore the idea, let's consider the following solution to the problem: I roll the big die and then take the value on the die that has position measurement with highest x value with respect to some grid on the table. If two dice have the same x positions, take the die with the highest y position. (Mathematicians call this lexicographic order.) That gives a very simple solution to the problem. A similar technique solves the bonus problem of simulating two six-sided dice.

However, that simple solution violates the indistinguishability requirement of the problem. We have distinguished the dice by choosing one of them. To understand the concept better, It's helpful to think of a way in which distinguishing the dice is impossible. Suppose you have a friend roll the big die in another room so that you can't see, then your friend sorts the numbers and reports them to you. So much information has been stripped out of the situation that you no longer have any capability to distinguish the three dice. The process may seem artificial, but it has many applications, for example in the study of elementary particles in physics, for which indistinguishability has major consequences.

So all we have is a sorted list of three numbers, like 1, 2, 5, or 3, 6, 6, or 5, 5, 5. How can we use the list to generate a uniform random number from 1 to 6? In line with the theme of getting our hands dirty, why don't we just list all the possibilities?

1,1,1
1,1,2; 1,2,1; 2,1,1
1,1,3; 1,3,1; 3,1,1
1,1,4; 1,4,1; 4,1,1
1,1,5; 1,5,1; 5,1,1
1,1,6; 1,6,1; 6,1,1
1,2,2; 2,1,2; 2,2,1
1,2,3; 1,3,2; 2,1,3; 2,3,1; 3,1,2; 3,2,1
1,2,4; 1,4,2; 2,1,4; 2,4,1; 4,1,2; 4,2,1
1,2,5; 1,5,2; 2,1,5; 2,5,1; 5,1,2; 5,2,1

and so on. I'm just writing the set of all triples of three numbers in lexicographic order, and then grouping equivalent (indistinguishable) triples together. I could just drop the second, third, and so on members of indistinguishable sets, but retaining them helps to remind me that the probability weight of getting certain outcomes may be different. For example, the probability of getting the sorted set 1,1,1 is 1/216, while the probability of getting the sorted set 1,1,2 is 3/216 because it can be generated by the dice in the three different ways listed,

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[4] *Adapted from* Parker, M. [standupmaths]. (2016, April 12). *The three indistinguishable dice puzzle.* Retrieved from https://youtu.be/xHh0ui5mi_E

and the probability of getting the sorted set 1,2,3 is 6/216 because it can be generated by the dice in the six different ways listed.

For space reasons, I've written just over 16% of the possibilities above, but if you're serious about getting your hands dirty, there's nothing wrong with writing them all out. It should only take a few minutes (or less, with an appropriate computer program). A word processor or spreadsheet can help organize your work. Once you have a list, you just have to assign the top sixth of the list to outcome 1, the second sixth of the list to outcome 2, and so on.

A good question to ask at this point is, how can we be sure that the list divides cleanly into sixths? Well, the truth is, it doesn't, as we can see from the top part of the list that I've written above. Including the first 9 lines has weight 31/216, and including the first 10 lines has weight 37/216, so we've missed the 36/216 mark which is required to take off the top 1/6 of the list. We can't split the 1,2,5 line because the six outcomes are indistinguishable.

How can we fix that? The problem is clearly the lone 1,1,1 at the top of the list. We have to group it with other outcomes to make a line with weight that will match that of other lines, so we have a nice even split. For example, we could group lines like this:

1,1,1; 2,2,2; 3,3,3; 4,4,4; 5,5,5; 6,6,6
1,1,2; 1,2,1; 2,1,1; 1,1,3; 1,3,1; 3,1,1
1,1,4; 1,4,1; 4,1,1; 1,1,5; 1,5,1; 5,1,1
1,1,6; 1,6,1; 6,1,1; 1,2,2; 2,1,2; 2,2,1
1,2,3; 1,3,2; 2,1,3; 2,3,1; 3,1,2; 3,2,1
1,2,4; 1,4,2; 2,1,4; 2,4,1; 4,1,2; 4,2,1
1,2,5; 1,5,2; 2,1,5; 2,5,1; 5,1,2; 5,2,1

Then each line contains all indistinguishable variations of an outcome and each line has weight 6/216. The first six lines have weight 36/216 = 1/6, so they can be mapped to the final outcome 1. The next six lines starting at 1,2,5 can be mapped to the final outcome 2, and so on. We have solved the problem. As a bonus, note that there are 36 lines, each of which can be mapped to a roll of two (distinguishable) dice, so we have solved the bonus problem too!

As far as solving the problem goes, we are done. However, the list above, in lexicographic order (roughly speaking), is hard to work with if you're not a computer. So your final challenge for this month's solutions is to find a way of solving the problem that is more amenable to human calculation. For example, is there a way of doing arithmetic on the three numbers to simulate a single die?

Something else you should consider is whether there's a way of simulating a single die with *two* indistinguishable dice, a simpler problem that maybe we should have started with.

For the bonus problem, there is no way of doing simple arithmetic on the three numbers to give you an answer, as far as I know. However, you should be able to find a way of combining lines on my original list and then ordering them so that there is a humanly comprehensible process to map the 36 resulting lines to pairs of outcomes for two distinguishable dice. You may decide that putting 1,1,2 etc. together with 1,1,3 is not as helpful as, say, putting 1,1,2 together with 4,4,5, for example, and then find a better ordering of the resulting 36 lines. The result should be a process that isn't hard to remember or execute. You don't want to have to walk around with a card in your wallet with long lists and/or a flow chart just to solve this problem. Ideally, you'll be able to develop a process with simple instructions which can be easily followed.

Edward Doolittle is Associate Professor of Mathematics at First Nations University of Canada. He is Mohawk from Six Nations in southern Ontario. He earned his PhD in pure mathematics at the University of Toronto in 1997. Among his many interests in mathematics are mathematical problem solving, applications of mathematics, and Indigenous mathematics and math education. He is also a champion pi-day debater at the University of Regina's annual pi day, taking the side of the other transcendental number, e.

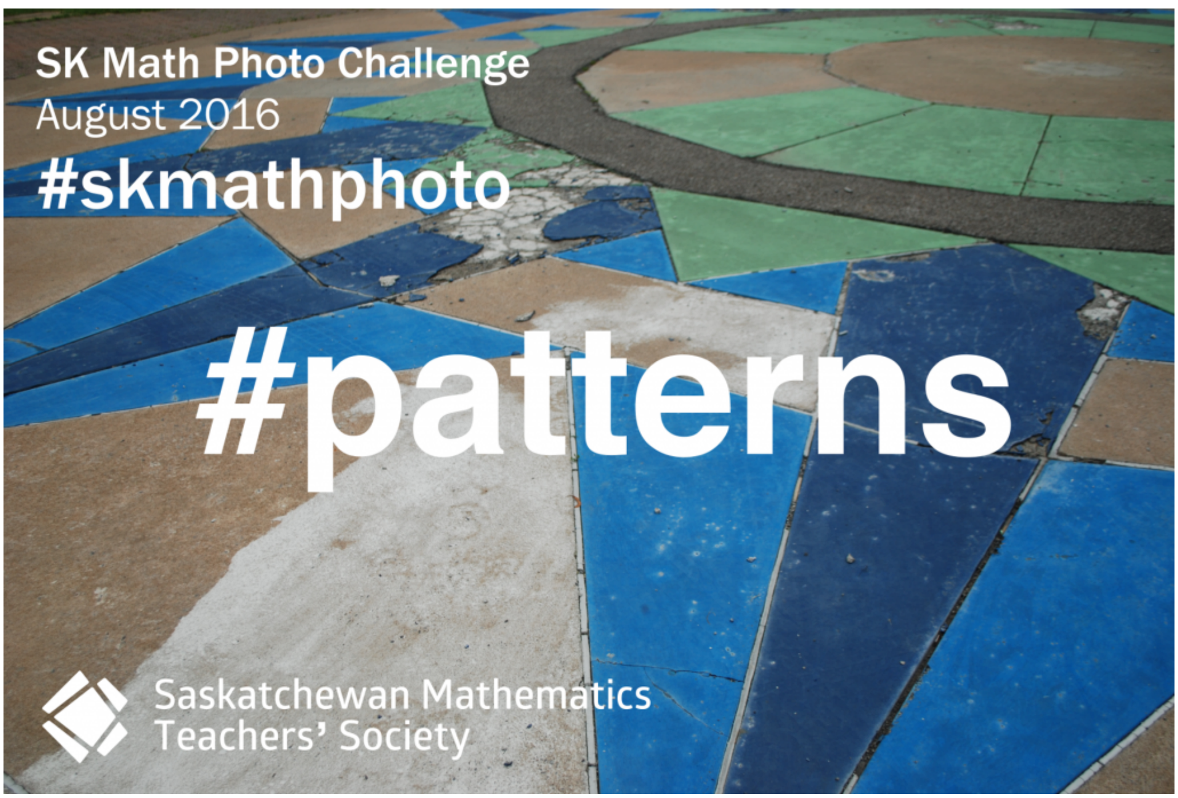
Saskatchewan Math Photo Challenge
August: Patterns

> What if I told you that you don't have to sail across an ocean or fly into space to discover the wonders of the world? They are right here, intertwined with our present reality.
>
> —E. Frankel, *Love & Math*

Inspired by the 2016 Math Photo Challenge (see the official website or the #mathphoto16 hashtag on Twitter), which revealed—through hundreds of submitted photos—the (mathematical) wonders all around us, we are continuing the fun with our very own Saskatchewan Math Photo Challenge!

Every month, we will choose a mathematical theme for you to explore in your photos. Keep your eyes peeled as you work and play, take photos of what you find, and share them on Twitter or Instagram using the #skmathphoto and current theme hashtags. See all photos submitted to the challenge on our website. At the end of the month, we will feature a few of our favorites here, in The Variable. Participation is *not* limited to math teachers, so encourage your friends, family, and students to play along!

This month's theme was…



#skmathphoto #patterns

Featuring photos by Nat Banting, Jennifer Brokofsky, Ilona Vashchyshyn, and Dean Vendramin

Reflections

Reflections *is a monthly column for teachers, by teachers on topics of interest to mathematics educators: lesson plans, book/resource reviews, reflections on classroom experiences, and more. If you are interested in sharing your own ideas with mathematics educators in the province (and beyond), consider contributing to this column! Contact us at thevariable@smts.ca.*

From Twitter to Twin Cities

Sharon Harvey

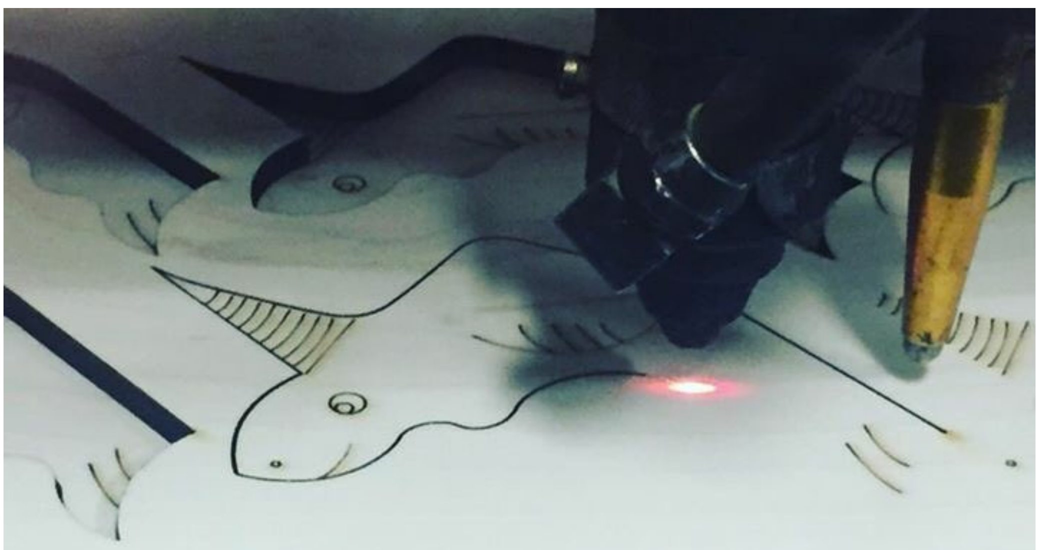
I'm convinced that my best learning happens when I listen to other teachers – when I hear their stories, try their strategies, and make lasting connections with them. This is exactly what Twitter Math Camp (TMC) is all about: It brings together 200 math/STEM educators from around North America to share with each other. It's professional development for teachers, by teachers. To learn more about TMC, check out the website http://www.twittermathcamp.com/, as well as this archived post from the first TMC held in July 2012: http://www.twittermathcamp.com/tmc-archive/tmc2012/genesis-of-tmc2012/ This year's camp was held at Augsburg College in Minneapolis, MN from July 16-19.

But I don't want my post to be about explaining this awesome opportunity. Rather, I'd like to share with you my favorite things about this year's TMC:

1. *Continuing sessions*. TMC organizes its 3 days of sessions so that your morning session is continued each day. This means that I spent 6 hours (essentially a day session) learning with the same session attendees and session leaders. The benefit of this over a single day session is that between sessions, I had time to process the new learning and solidify it before moving on. It also meant that I was able to bring learning from my afternoon sessions back to my morning session.

   I chose to attend a session called Tessellation Nation and spent three days with Christopher Danielson (a favorite from the 2016 Saskatchewan Understands Math conference) exploring all my questions about tessellations. This session also included a trip to the maker space where Christopher cuts his famous turtles and narwhals.

Exploring tesselations at the Tesselation Nation sessions

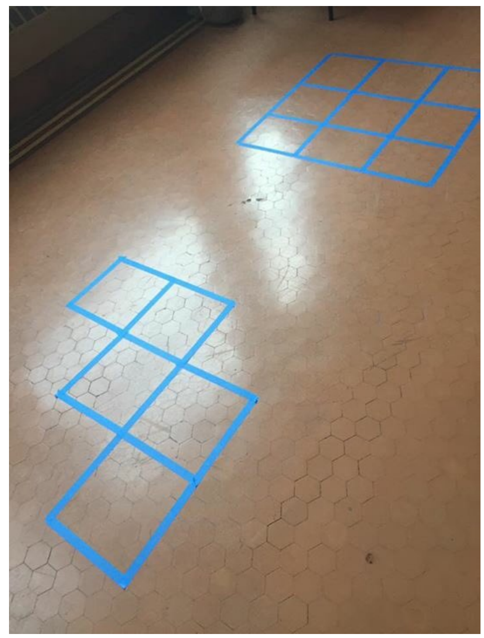


2. *"My Favorites" sharing.* Each day, several teachers presented during mini-sessions called "My Favorites." They took place following opening announcements or prior to the keynotes, when we were all gathered as a large group. "My Favorites" had teachers sharing a favorite of theirs—a lesson, a website, a teaching strategy—all in 15 minutes or less. This means that, in addition to the 10 sessions I attended, I also got to see 20 teachers share something they cared about. Many of the favorites have already been added to my bag of tricks for next year. I am excited to use visually random groupings, high fives, and maps to extend my students' learning.

3. *It gives bodies to the handles.* It's called Twitter Math Camp because during the rest of the year, we spend our time on Twitter as part of the Math Twitter Blog-o-Sphere (MTBoS), supporting and challenging each other. Through the social network, we share our ideas, our blogs, and generally belong to each other. TMC allowed me to meet the people who developed the lessons I've been using, or the websites I share with kids, or who are

willing to answer any question I ask. If you want to know more about MTBoS and why you should join, too, check this website out: https://exploremtbos.wordpress.com/. MTBoS is for EVERYONE!

4. *Conversations*. The conversations were amazing and very welcoming – everybody's in at TMC. Simply being there makes you part of the conversation. I spent 3 full days learning how to support my kids—and not once did we talk about what our students *couldn't* do, nor did we focus on what's wrong with the curriculum. Instead, we focused on getting students where we need them to be. We talked about what we needed for that to happen, and then we got started on actively doing it. I didn't sit through sessions and then simply tuck that information away for later use.



5. *It's CAMP*. I spent 5 days sleeping in dorms with 200 other people. We ate together, we learned together, and we played together. The evenings were spent together playing trivia at a local pub, hiking to a waterfall together, hitting up the Mall of America, and even learning to ride a backwards bicycle. New friendships developed quickly and old ones were deepened. And, as you can see in the photo on the right, there was ample floor space in a dorm, enough for Malke Rosenfeld to help us all explore 'math in our feet'! If you haven't heard of Math In Your Feet, check it out here: http://www.malkerosenfeld.com/math-in-your-feet-for-students.html

If you didn't attend TMC16, you can still do some of the learning that I did. TMC has a wiki where many of the presenters have posted their session outlines, PowerPoints, and other resources (see http://twittermathcamp.pbworks.com/w/). You can also explore the TMC archives from previous years on the TMC website.

And finally, you can get started on planning for your attendance at TMC 2017 in Atlanta, Georgia! I hope to see you there.

Sharon Harvey has been a teacher within the Saskatoon Public School Division for eight years. She has taught all secondary levels of mathematics, as well as within the resource program. She strives to create an inclusive and safe environment for her students.

Spotlight on the Profession
In conversation with Grace Kelemanik

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Grace Kelemanik, who will be presenting at this year's Saskatchewan Understands Mathematics (SUM) Conference in Saskatoon.

Grace Kelemanik works as a mathematics consultant to districts and schools grappling with issues related to quality implementation of the Common Core State Standards. She is particularly concerned with engaging special populations, including English Language Learners and students with learning disabilities, in the mathematical thinking and reasoning embodied in the eight Common Core standards for mathematical practice.

Kelemanik is a secondary mathematics Clinical Teacher Educator for the Boston Teacher Residency Program, a four-year teacher education program based in the Boston Public School district that combines a year-long teacher residency in a school with three years of aligned new teacher support. Prior to BTR, Grace was a project director at Education Development Center (EDC). She was lead teacher of mathematics at City on a Hill Public Charter School in Boston where she also served as a mentor to teaching fellows and ran a support program for new teachers. Grace is co-author of the book, Routines for Reasoning, about instructional routines that develop mathematical practices.

I would like to begin by asking you a little bit about your background. Could you describe your journey to teaching mathematics, and then teaching future mathematics teachers? What (or who) sparked your passion for the field of mathematics education?

My mother will tell you that she always knew I would become a math teacher. She reminded me of this when, after entering college as a music therapy major and exiting with a degree in finance, I decided to go to graduate school to study mathematics and education. She said she always knew when there was a math test, because our phone would "ring off the hook" and she would listen while I spent countless hours explaining math concepts to my classmates. If she had shared this insight with me earlier, I would have come to math teaching sooner, but then I never would have met Mark Driscoll.

It is because of Mark Driscoll that I am a teacher educator. When I moved to Boston and started grad school at Boston College, I was in desperate need of a job. Mark, a project director at Education Development Center (EDC), was looking for research assistant to work on his newly funded Urban Math Collaborative project, and he hired me. I am still not sure why he took a chance on me, but that decision changed the trajectory of my life. Working with Mark (and others at EDC), I became steeped in the math reform movement of the 80s. Working in urban settings, I became acutely aware of the disparity in our education system. It was clear to me that great teachers – even those working with limited resources—could have a profound impact on students. My compass was set. I would become an urban math teacher, and then a teacher educator.

A recent report by the U.S. Department of Education's National Center for Educational Statistics (Gray, Taie, & O'Rear, 2015) suggests that as many as 17% of new teachers in the United States may be leaving the profession within their first five years on the job. Some estimates suggest that the figure may be similar in Canada, although the relevant data is scarce (Karsenti & Collin, 2013).

In your current position, you work with new teachers as a secondary mathematics Clinical Teacher Educator for the Boston Teacher Residency Program (BTR). [The BTR is a four-year teacher education program based in the Boston Public School district that combines a year-long teacher residency in a school with three years of aligned new teacher support.] In your view, why is the teacher turnover so high (and why might this be a worrisome state of affairs)? What kind of support do you feel new mathematics teachers need during their first few years of teaching to increase the chance that those with great potential do not leave the profession?

You noted a US Department of Education statistic that 17% of new teachers in the US leave the profession within their first five years. The statistics for urban teachers are even more grim, with 50% of all urban school teachers leaving within the first three years.

Teaching is a hard job to do well. It requires deep content knowledge and a strong belief that all students can learn. But teaching is not magic, nor is it an art form—teaching is a learnable practice. Developing a practice takes time and support. Unfortunately, time and support are more often than not in short supply for novice teachers. We have expected novice teachers to develop ambitious teaching practices by spending a relatively large amount of time studying the theory of teaching, but a relatively little amount of time actually practicing teaching. What is more, the limited targeted teaching support we provide teacher candidates all but dries up when they become teachers of record. So it is no surprise that the hard work of teaching becomes crushingly hard in the first few years, as too many underprepared teachers struggle—all too often on their own—to get their teaching "legs".

"Teaching is a hard job to do well. It requires deep content knowledge and a strong belief that all students can learn. But teaching is not magic, nor is it an art form—teaching is a learnable practice."

Teaching, as Magdalene Lampert says, is a complex endeavour. It requires you to attend to a multitude of things at the same time, including the content you are teaching, how students are making sense of that content, interactions between students, the flow of the lesson, individual student learning needs, how to manage classroom materials, and more. For a beginning teacher, this requires an especially large "bandwidth," and can be overwhelming. Accordingly, a critical support for beginning teachers is something that helps them manage the complexities of teaching. We have found that Instructional Activities Structures (IAs) or instructional routines fit that bill. In the BTR program, we have used IAs to help "routinize" classroom interactions. These instructional routines bring a predictable flow to a lesson so that the teacher (and students!) can spend less time worrying about what's coming next and can use more of their bandwidth listening to student ideas and helping them make sense of the mathematics. The instructional routines also simplify lesson planning, because they hold the design of the lesson constant in terms of how students will interact with the content and with each other. This allows the novice

> "If new teachers weave instructional routines into their practice, it will provide them some solid ground on which to stand while they continue to build their practice."

teacher to focus their planning time and energy on the mathematics and how their students will make sense of the math. If new teachers weave instructional routines into their practice, it will provide them some solid ground on which to stand while they continue to build their practice.

Instructional routines are also a powerful tool for collaboration. When entire departments or grade levels use the same instructional routines, it provides a common frame for collaborative lesson planning. Because every teacher knows "how the lesson is going to go," they can jump right in to discussing the mathematics, anticipating student responses, and supporting individual student learning needs. Instructional routines support the work of math coaches for the very same reason—the routines hold the structure of the lesson constant so that the teacher and coach can focus on the critical interactions between the students and the content.

Over the years, I have watched as these instructional routines have become like "old friends" that our BTR grads bring into the classroom with them to lean on while they take on the hard work of teaching.

With respect to students' needs, you are particularly concerned with engaging special populations, including English language learners, in the mathematical thinking and reasoning embodied in the Common Core standards for mathematical practice. Although some see mathematics as a "universal" language, it is clear that English language learners (or "emergent bilinguals," the term advocated by Rochelle Guitierrez in her most recent ShadowCon session) need additional, or different kinds of support in the mathematics classroom. How are these students' needs different, and how can their teachers support them in learning mathematics in a non-native language?

You are also particularly concerned with helping students with disabilities succeed in mathematics. It is, of course, difficult to generalize, as students with disabilities have distinct and individual needs, but could you describe your philosophy about engaging and supporting such students in the mathematics classroom?

I am going to answer your questions about English learners and students with disabilities together, because it turns out that there is a great deal of overlap between the research-based supports for English language learners and those for students with learning disabilities. Both groups benefit from mathematics being placed in authentic, meaningful contexts to which they can relate, the use of multimodal techniques, regular opportunities for language use, and scaffolds for increasingly abstract thinking. I'm a firm believer of teachers focusing on the overlap, not just because these same supports will help a wider range of learners, but it also turns out—and this is quite powerful—that these supports also align with the approaches to doing math championed in the Common Core state standards for mathematical practice! As my colleague and coauthor Amy Lucenta says (See Amy's 2016 NCSM Ignite Talk at https://youtu.be/M6MTNzs4J44), "There is a symbiotic relationship between the math practices and supports for special populations. If we teach the math practices authentically, we'll be supporting special populations, and if we support special populations with integrity, we'll be teaching the math practices."

In your sessions with mathematics teachers (e.g., during your Ignite talk at the 2015 NCTM Annual Meeting and Exposition – https://youtu.be/oTmYi1Gsa70), you stress the importance of mathematical practices and of guiding students in learning to "think like mathematicians." Could you give an overview of the mathematical practices advocated by the Common Core State Standards (and the NCTM) for our Canadian mathematics teacher audience, and why you advocate focusing on practices rather than (exclusively) on content?

> "We can't "algorithm" our students into being ready to solve the problems they will face as adults. So we have to teach them how to think."

I believe that teaching students to think like mathematicians is critical because, as Al Cuoco, a colleague of mine from EDC is fond of saying, "The technologies that are going to create the problems that our students are going to have to solve have not yet been invented." This means that we can't "algorithm" our students into being ready to solve the problems they will face as adults. So we have to teach them how to think. The CCSS math practice standards, I believe, is our clearest articulation to date of what it means to think (and work) like a mathematician. Therefore, we must teach the practices!

In the Ignite talk you referenced and in our book, *Routines for Reasoning Fostering the Mathematical Practice in ALL Students* (Kelemanik, Lucenta, & Creighton, in press), my coauthors and I argue that not all math practices are equal—that some take the lead in student thinking, while others support that thinking. We argue that three math practices in particular, *Reason Abstractly and Quantitatively (MP2), Look and Make Use of Structure (MP7), and Look for and Express Regularity in Repeated Reasoning (MP8)* define three distinct ways of thinking mathematically. We believe that that if students develop these avenues of thinking, they will become powerful problem solvers.

Our readers will likely be aware that you will in Saskatoon this November to present as a keynote speaker at our very own Saskatchewan Understands Math (SUM) Conference. (We can't wait!) We don't want to spoil the punchline, but could you give our readers some insight into what you will be discussing during your sessions?

Sure! My plan is to share with you our framework for making sense of the standards for mathematical practice, to unpack the three avenues of thinking, and to introduce folks to how instructional routines can be leveraged to develop these practices in all students, including English language learners and students with learning disabilities.

Thank you, Grace, for taking the time to share your experiences and your expertise. We look forward to continuing the conversation at SUM in November!

Ilona Vashchyshyn

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Encouraging Mathematical Habits of Mind: Puzzles and Games for the Classroom
Hidato and Latin Squares

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www.susansmathgames.ca

Hello again! This time I'd like to share two types of puzzles that appeal to people of all ages. They are of very different types, one focusing on number and the other on attribute. They're good to introduce at the beginning of a school year, as their rules are easy to understand and they develop different types of problem-solving skills.

For a brief discussion of the overall benefits of playing math/logic puzzles and games in the K-12 classroom, please see my previous article in the June 2016 edition of *The Variable*.

Templates for all of the puzzles I'll describe are available at my website, along with descriptions of the methods that I've found to be effective for introducing puzzles to a class.

Hidato

Although the name sounds Japanese, this game was invented by an Israeli mathematician, Dr. Gyora M. Benedek. "Hidato" means "riddle" in Hebrew.

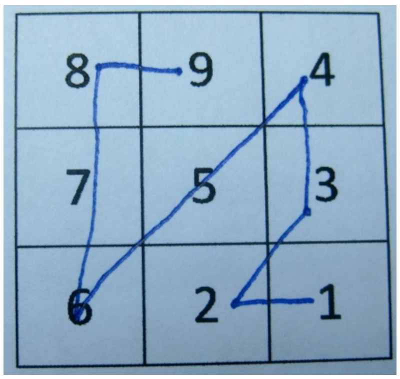
I introduce the game to all grades by asking what the students notice about a **solved** puzzle:

Take a moment to look at it yourself!

8	9	4
7	5	3
6	2	1

Observations usually include most of these comments, in roughly the following sequence:
- there are all the numbers from 1 to 9
- they are not in the usual order
- it's sort of like Sudoku but not really, because there are only 9 little squares
- the middle row is all odd numbers, but there isn't anything like that for any other row or column (a really good example of an observation that turns out to be a "non-rule")
- the numbers 6, 7, and 8 form a straight line up the left side (I draw those lines)

- aha! at this point, someone notices that 9 is connected to 8, and 1 is connected to 2 (I draw those lines)
- aha again! someone points out that 2 is connected to 3 by a diagonal line, and then the suggestions come thick and fast, and it takes no time to draw in the lines that make it clear to everyone



In order to encourage the use of precise language, I always ask the class about the types of lines. Interestingly, even the youngest students can identify a slanting line as "diagonal," but many in elementary school refer to the other two types as just "straight." Those who haven't run into the words "horizontal" and "vertical" seem happy to learn them.

The rule: Each counting number needs to be directly connected to the subsequent one by a horizontal, vertical, or diagonal line.

Now I can congratulate the class on having figured out the rule and ask if they would like to solve a puzzle together. I'll have younger grades look at a 3x3 example, intermediate grades look at a 4x4 example, and high school students look at a 5x5 example. We usually solve only one together, unless the group is keen to try another, harder puzzle as a class. Then I turn them loose to work on a sheet of graduated puzzles on their own.

See if you can solve all three of these! Each has only one solution. You don't need to draw in the connecting lines, but doing so can help you be sure that you have solved the puzzle.

9		6
3		
	2	**1**

		10	
8	6	**1**	
			16
3	4		14

25	4	23		
5			22	19
	11		**1**	
7	10			17
		14	15	

With all of these, it is common for someone to suggest an erroneous choice partway through. I'm happy if it happens, as it provides a good demonstration of mathematical thinking: we have to find the source of the error and erase it. We learn something from our errors, too. I've observed in scores of classrooms that no one gets upset by this – it's only a game, after all!

As an example, suppose that we place the 4 beside the 3 in the 3x3 puzzle.

9		6
3	4	
	2	**1**

We can continue the puzzle as far as possible, if the students choose to do so, producing something like this:

9	7	6
3	4	5
	2	**1**

Usually before this point, students will be clamouring to tell me that something is wrong. Our choice of the place to put the 4 has created a hole that cannot be filled. It's good to watch out for these holes!

9		6
3	4	
	2	**1**

In the 4x4 puzzle, it's easy to place the 2, but there are two logical possibilities for the 5, indicated with a * below:

		10	
8	6	**1**	
*	2	*	**16**
3	4		14

Some students will choose the correct place accidentally, some will choose the incorrect one, and some will think it through and be able to explain their logic to the class.

I always hope for the incorrect choice (the * on the right), so that we can discuss it!

I encourage the class to pay attention if ever they reach a spot where there are two logical possibilities so that, if they find that they get into a mess, they need to erase everything only as far back as that branch point.

The 5x5 puzzle might look a bit intimidating, but if you take a moment to consider the 25, you'll realise that there is only one place to put the 24. That means there isn't any choice for the 3 and the 2. The rest falls into place easily.

This illustrates a very useful technique for Hidato puzzles of any size: sometimes, it is more helpful to count backwards than forwards!

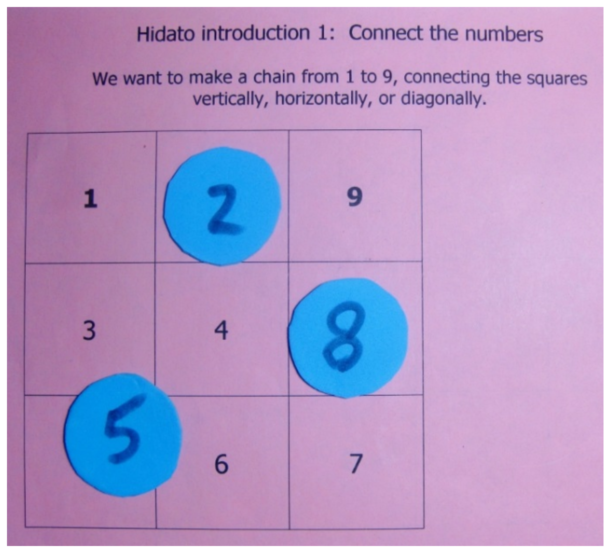
25	4	23		
5	24	3	22	19
	11	2	**1**	
7	10			17
		14	15	

Hidato with Manipulatives

I find it very helpful to have K-2 students use manipulatives when playing Hidato, and often start out Grade 3s with them as well. It's easier and less intimidating for younger children to move pieces around than to have to erase.

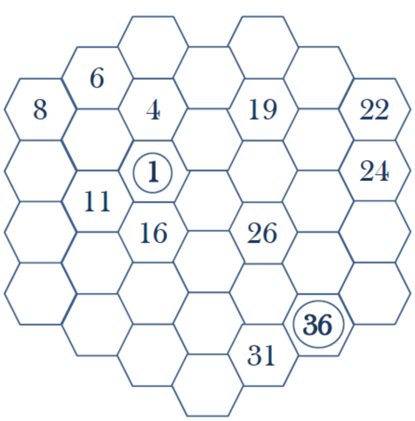
Some students start by covering the whole grid with the numbered discs. This occasionally results in them moving *all* of the discs about in an attempt to get the numbers in order, so I generally ask them to put in only the missing numbers.

These puzzles with manipulatives work their way up from size 3x3 to 5x4. Once they've solved these, children are keen to get into the puzzles for the "big kids."



Hexagonal Hidato

Recently, I've experimented with a hexagonal version of Hidato that looks more complicated, but actually involves fewer possible directions for the number chains. Many people say the hexagonal puzzles are easier than puzzles of the same size in the original version, although this may partly be the case because they've already played the original version and have developed some strategies for solving them.



Sources of puzzles

The website Math in English (www.mathinenglish.com) has a selection of printable Hidato puzzles for primary to intermediate grades. It also has a simpler version of the puzzle, called Numbrix, which involves only vertical and horizontal connections. I think the diagonal connection is what makes the puzzle fun, but your students might enjoy the Numbrix puzzles.

On my website, you'll find puzzles designed to be used with manipulatives, a larger set of graduated puzzles for older students, and some other materials that might be helpful.

Finally, for a wide variety of puzzles that include some serious challenges, go to the official Hidato website (www.hidato.com).

Latin Squares

This set of puzzles has much to offer:

- it is visually appealing, with bright colours and distinct shapes rather than numbers or letters;
- it requires puzzlers to be very clear about which rules they are using and about the precise meaning of the words used;
- there are hundreds, if not thousands of solutions for each level;
- there is lots of scope for students to create patterns within the given restrictions;
- solutions can be found via many approaches.

Many entertaining puzzles, including Sudoku, are based on Latin squares. There is a lot of beautiful mathematics associated with them as well.

I've found that there are some people (of any age) who really dislike the look of all of the numbers in Sudoku, to the point that they refuse to engage with it at all. However, even they will get drawn into a simpler puzzle that involves bright colours and strong shapes.

Once they've experienced and enjoyed the various challenges associated with these pieces, students are generally keen to get into other puzzles based on Latin squares, such as Kakurasu, Towers, Futoshiki, Neighbours, and Mathdoku. (I hope to describe all of these in future articles.)

While I originally introduced the idea to K-2 students using a 3x3 grid and three colours of discs, I have since found that they are just as happy as are older students with this 4x4 version, which uses four shapes in four colours and allows for many more variations.

It's fascinating just how widespread the appeal is of making patterns with these pieces!

Introductions

I introduce these puzzles to students of all ages by first mentioning that mathematics is about more than numbers – it is about finding and thinking about patterns. Human beings love playing with patterns! And mathematicians aren't the only ones who are fascinated by them: for example, musicians, visual, and fabric artists from all eras and nations also regularly work with patterns.

I give each student a bag with the pieces in the above photo. I ask students to figure out if any pieces are missing. I used to ask them to "sort the pieces," but then I'd get asked, "How do you want us to do it?" Figuring out if any pieces are missing seems to free the imagination of more students, so they become focused on the actual task rather than on trying to guess what I want.

What I really want is for students to realise that there may be several equally good ways of solving a problem. Sometimes, a student will indeed be missing a piece or perhaps have an extra piece, so having them check first is helpful in another way.

Once everyone is satisfied that they have the correct pieces, I ask how they know. The sophistication of the answer varies with grade level, but very few people seem to have any difficulty in deciding.

I also ask how the students sorted their pieces. How many made piles of the same shape? How many made piles of the same colour? How many organised their pieces in another way? Students can compare their method to that of their neighbours.

I like to go over the names of the shapes, especially with elementary students, and point out that the orientation of a shape doesn't change its name. Recognising this is part of a child's mathematical development. The square causes the most discussion, as students tend to call it a "diamond" when it is standing on a point. We've had quite a good discussion about this in some classes.

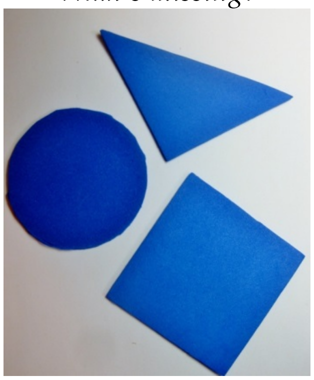
Another source of discussion is the fact that the smallest piece has a special mathematical name, which is *not* "diamond." Diamonds come in many different shapes, almost none of which look like this:

I tell the class that mathematicians refer to this shape as "rhombus," which means that all four sides are of the same length. Often, someone will then point out that a square must therefore be a rhombus. How far we go with a discussion about names and characteristics depends on the grade and the interest shown by the class. At the very least, I tell them that I'll know what they mean if they call something a diamond, but if they want to impress people by knowing the technical term, now they can.

Fun for the Youngest

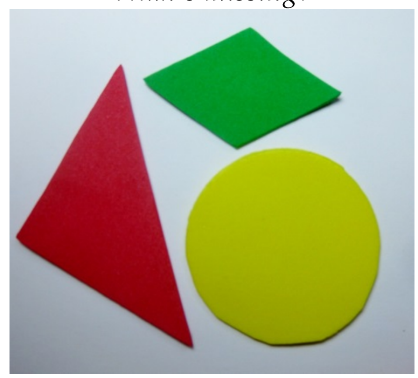
Before going on to the grid puzzles, I've found that K-2 students really enjoy playing a short game based on attribute. Here are some examples. In each case, I ask them to find and hold up the piece that is missing from the set I am holding up.

What's missing?



What's missing?

What's missing?



A Graduated Set of Variations

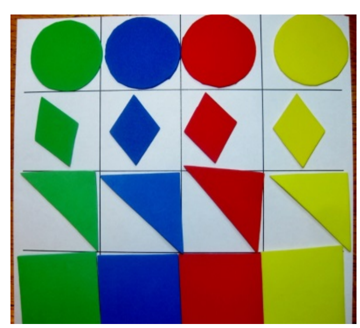
I introduce this set of puzzles to Grades 2 and above differently from the way I introduce most others, writing the rules – one level at a time – on the board, as the students complete each level. That way, students can double-check the rules that they are currently using. I think it is useful for students to see the progression in complexity, and for them to realise that the rules are what makes the game. If I don't put up the next rule quickly enough, students will often suggest it. (In grades where all students might not be able to read yet, we usually work through a few levels only, one at a time, with verbal direction only.)

Getting the idea across initially often requires several re-statements, as some people oversimplify and others make it more complicated than necessary. I have discovered that in any age group, there are people who can hear a rule and proceed to use it, others who can read it and proceed to use it, others who read a rule and need to re-state it in their own words before using it, and some who need to hear the re-statement from their peers.

Here are some of the variations I use. Where we start and end in the list depends on the grade and the amount of time available. More than once, I've been surprised by a class wanting to solve ever-harder variations, even going well into their recess time.

Level 1. *Every row should have four colours* ***and*** *every column should have four shapes.*

Stress the difference between rows and columns!

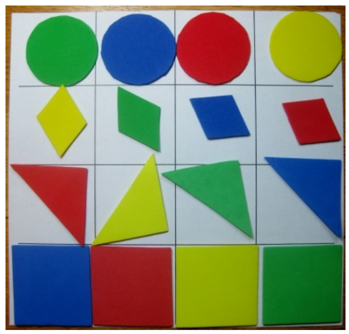


Variations:

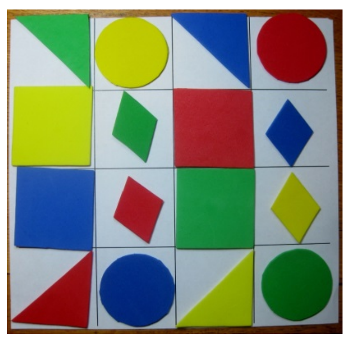
- Use different phrasing, such as "no colour should repeat in any row and no shape should repeat in any column."
- Put a similar completed pattern on the board and ask the class to describe the rule you used.

Level 2: A Latin square. *No colour should repeat in any row* ***or*** *in any column (every row should have four colours and every column should have four colours).*

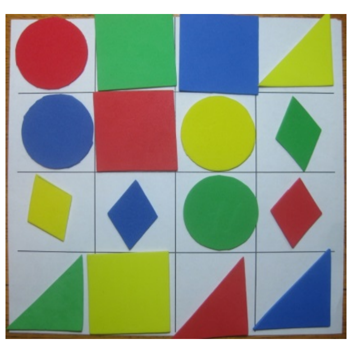
Here's an example made systematically from the first one. Can you figure out how?



Here's an example based on patterns within 2x2 blocks. I would never have thought of it, but students do!

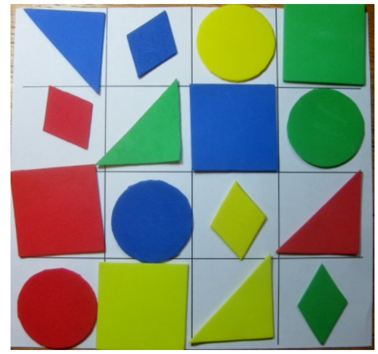


Here is an example which seems to have been made with nothing else in mind other than the stated rule.

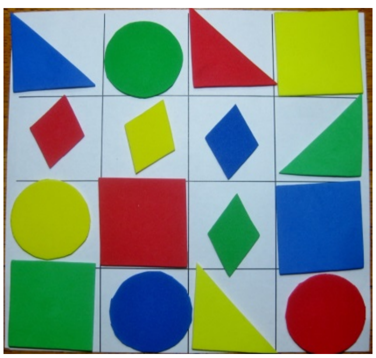


Level 3: A Latin square. *No shape repeats in any row or any column. (Ignore colour!)*

Most people say that this is a bit harder than the same problem with the restriction on colours instead of on shapes. It might be easier if the shapes were all the same colour, but otherwise it seems that generally, colour wins the competition between colour and shape.



Level 4: A Latin square with an extra restriction. *No colour repeats in any row, column,* ***or*** *along either of the two big diagonals. Note that it is impossible to avoid repetition in shorter diagonals, at least with a 4x4 puzzle.*



This presents a considerably greater challenge! I've found that adults have very little advantage over youngsters when it comes to this level. It's nice to have a puzzle that puts everyone in the same boat.

I like to let students try this for a while on their own. Some will find a solution fairly quickly – I ask them what their strategy was (if any), then I might ask them to find a different solution or quietly suggest that they go on to the next level, which is to consider shape instead of colour.

After a few minutes, I suggest to the class at large that they might consider starting with the diagonals, as those are the trickiest to see.

After we've had a number of solutions turn up, I've found it helpful to get the attention of the whole class and work through a likely strategy. There are several. One, suggested to me by a Grade 4 boy in a First Nations school, is to start with the corners, then put in the diagonals. I draw a grid on the board and ask for suggestions on how to go around the four corners. Clearly there are a number of possibilities. Then, we fill in one of the diagonals. There are only two possibilities for this. The second diagonal can be filled in only two ways, one of which might lead to problems, depending on what you've done with the first diagonal. Immediately below, I've illustrated a set of choices that will produce a solution.

Fill in the rest to see it!

R			B
G			Y

R			B
	B		
		G	
G			Y

R			B
	B	R	
	Y	G	
G			Y

Depending on the curiosity of your class and their suggestions along the way, you might find yourself discussing what to do if you've created a situation that won't work, as below.

R			B
	G		
		B	
G			Y

R			B
	G	R	
	Y	B	
G			Y

R			B
	G	R	**??**
	Y	B	
G			Y

There is nothing we can put in the square with the **??**. *However*, we needn't start all over again. Let's just switch the position of the two colours we put in last, red and yellow. Now we can solve the puzzle.

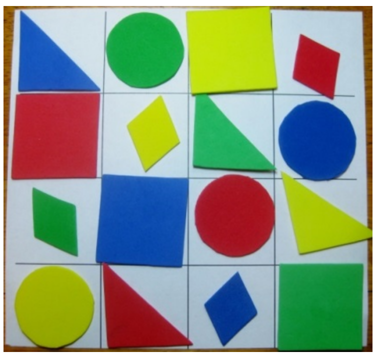
R			B
	G	Y	
	R	B	
G			Y

In all logic games, there is an essential difference between moves that we are required to make and those involving a choice. If we make a choice, we need to be aware that we have done so. If something goes wrong down the line, it's often enough to go back to our last choice and change it, rather than start all over again. This doesn't always work, but it is one more tool in the problem-solving kit. I like to illustrate it in class whenever possible.

Level 5: Another Latin square with diagonals. *No shape repeats in any row, column, or along either of the two big diagonals.*

We don't always do this one, but it is nice if there is time. Students can try the strategy described above, if they like, using shapes instead of colours.

Level 6: The big finish



Can you figure out the rule?

No shape ***or*** *colour repeats in any row, column, or along either of the two big diagonals.*

This is an example of a mutually orthogonal Latin square (whew!). It is also called a Graeco-Latin square, because Leonhard Euler, who studied them in the 1700s, used Latin and Greek letters instead of colour and shape. I much prefer to call it an Euler square, however, as that is a shorter name and honours a very famous mathematician. (By the way, "Euler" is pronounced "Oiler.")

Further fun and games

Students who have really enjoyed these 4x4 puzzles might like to try the 5x5 equivalents. It's possible to avoid repetition along **any** diagonal in a 5x5 grid. It isn't possible in a 6x6 grid, but it is possible in a 7x7 grid. Apparently, it's possible to avoid repetition along any diagonal in an *n*x*n* grid as long as *n* is not divisible by 2 or by 3*.

Older students might enjoy finding out about some the mathematics associated with Latin squares.

Materials

To make long-lasting sets of pieces, I use coloured plastic foam, which I find at craft stores, dollar stores, and some office supply stores.

An early version of this used playing cards, for example the jack, queen, king, and ace of each of the four suites. While I find that the colour-shape pairing is easier to see, I have had some fun using a deck of cards when sharing the puzzle with friends.

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* This is related to the fact that one way to solve a Latin square is to use the knight's move from chess (e.g. 2 over and 3 down) to place the next piece of the same colour. Imagine your grid as being wrapped around a cylinder with the left and right edges touching in order to cope with going off the edge.

To quickly make a visually appealing game for intermediate students at a science camp, I made four squares of each of four colours of paper and wrote the numbers from 1 through 4 on each colour. This was actually the first time I had tried the game out, and I was delighted by how much fun the campers had with it.

And that's it for now. I'd love to hear from you about your experiences with either of these puzzles! There is a "contact me" form on my website, www.susansmathgames.ca

Susan Milner taught post-secondary mathematics in British Columbia for 29 years. For 11 years, she organised the University of the Fraser Valley's secondary math contest – her favourite part was coming up with post-contest activities for the participants. In 2009 she started Math Mania evenings for local youngsters, parents, and teachers. This was so much fun that she devoted her sabbatical year to adapting math/logic puzzles and taking them into K-12 classrooms. Now retired and living in Nelson, BC, she is still busy travelling to classrooms and giving professional development workshops. In 2014 she was awarded the Pacific Institute for the Mathematical Sciences (PIMS) Education Prize.

Teaching Math Holistically in Our Classroom [5]

Anamaria Ralph

> *Mathematics instruction is most effective when it is part of a comprehensive approach that pays equal attention to all aspects of a child's development: social/emotional, physical, cognitive, and language. There must be opportunities for children to explore independently and times for direct teaching in small groups, in large groups, and one-on-one. In addition, children practice and use mathematical skills during daily routines and during the active learning that takes place in interest areas.*
>
> (Copley, Jones, & Dighe, 2007, p. ix)

My teaching partner and I have reflected for quite some time on how to create a math program for young children that is grounded in a holistic approach to learning. This year, we have found what seems to be working well for us with this particular group of children. Given our schedule, the environment, and the children, this is our example of how math can be incorporated into the daily Kindergarten program.

Direct Instruction (Small Groups)

Instruction is tailored to meet the needs of small groups of children.

Educators provide experiences in playing with mathematics itself by using a repertoire of strategies, including open and parallel tasks that provide differentiation to meet the needs of all students and ensure full participation. (Ontario Literacy and Numeracy Secretariat, 2011, p. 2)

[5] A prior version of this article was published on December 7, 2015 on Anamaria's blog, *Wonders in Kindergarten* (http://wondersinkindergarten.blogspot.ca/). Reprinted with permission.

Provocations

Provocations are set up based on small group direct instruction content. They give students the freedom to explore, investigate, and experiment with the materials at their own level of understanding.

Honouring children's starting points enables educators to build on students' mathematical knowledge with an inquiry-based approach, developing purposeful and meaningful mathematical experiences in the classroom. (Ontario Literacy and Numeracy Secretariat, 2011, p. 2)

"What number stories can you tell?"

"Can you guess the number?"

"Roll the dice! Record on the 10 frame!"

"How many letters are in your name? Who has the most? Who has the least?"

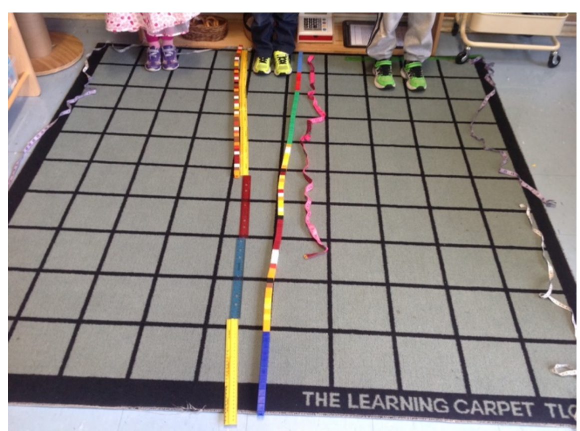
"What is a pattern? Where do you see patterns in real life? What different patterns can you make?"

"What do you notice?"

Reflection: Sharing of Work (Whole Group)

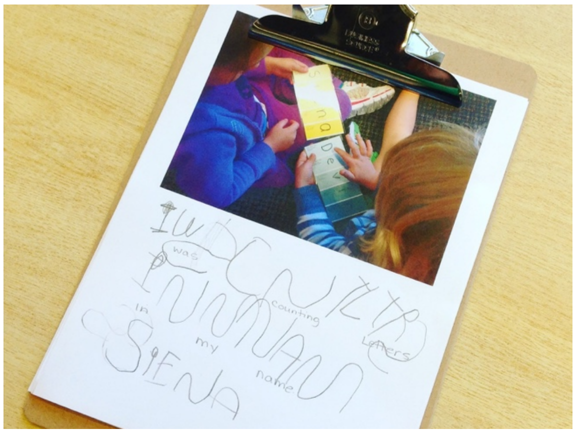
Students talk about their thinking with peers. New learning and extensions are made through questions and comments posed by peers and educators. This allows for whole-group sharing of knowledge and ideas.

After students have worked through solving a problem, educators facilitate consolidation time (either with individual students or with small groups or large groups) in order to allow students to talk about their thinking. As educators value a variety of strategies and solutions, they guide students to make connections between them, to recognize how the thinking relates to the key mathematical concept and to make further conjectures and generalizations. (Ontario Literacy and Numeracy Secretariat, 2011, p. 4)



Another way to encourage further learning through sharing is through documentation. We have set up a Mathematicians wall above the Math Area with photographs capturing various student creations and learning experiences, which the students enjoy drawing inspiration from as they seek to challenge themselves further.

Teachers can post documentation of math learning as a way of encouraging children to reflect on past experiences and motivate them to plan and revise future ones. (McLennan, 2014, p. 21)



"I was counting letters in my name"

"We are mathematicians"

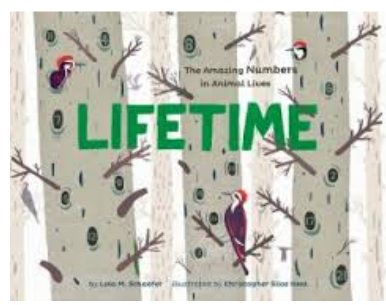
Math-Related Books

We use math-related books as another way to give students opportunities to apply emerging mathematical understandings. As Cutler, Gilkerson, Parrot, and Bowne write,

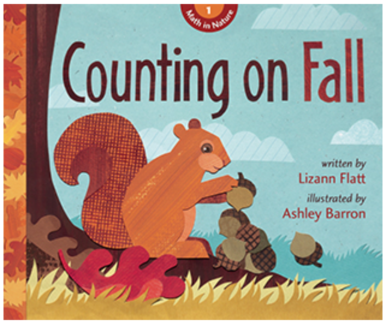
Integrating mathematics and literacy creates an interweaving of curriculum rather than a compartmentalizing of academic subjects. Many children's books provide a natural, meaningful path for exploring and exchanging ideas about math concepts. (2003, p. 20)

These Bees Count!
Alison Formento, illustrated by Sarah Snow

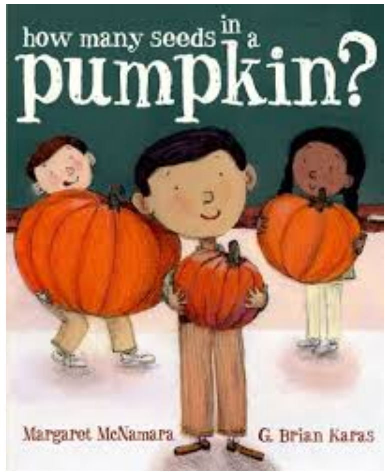
Spotty, Stripy, Swirly: What Are Patterns?
Jane Brocket



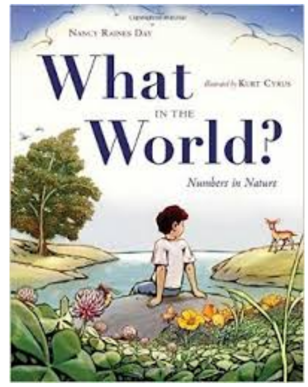
Lifetime: The Amazing Numbers in Animal Lives
Lola M. Schaefer, illustrated by Christopher Silas Neal



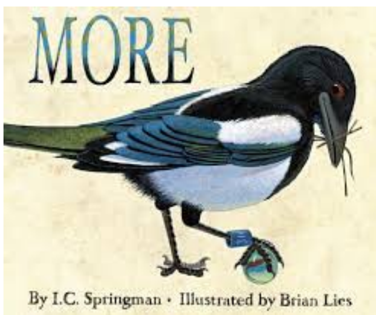
Counting on Fall
L. Flatt, A. Barron)



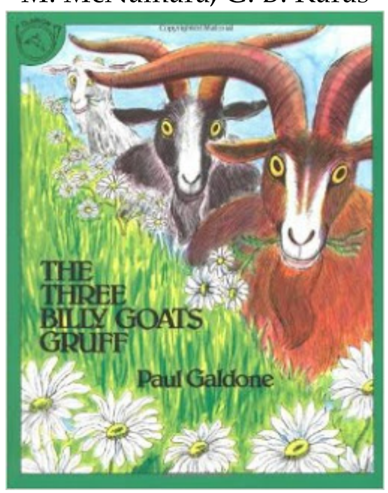
How Many Seeds in a Pumpkin?
M. McNamara, G. B. Karas



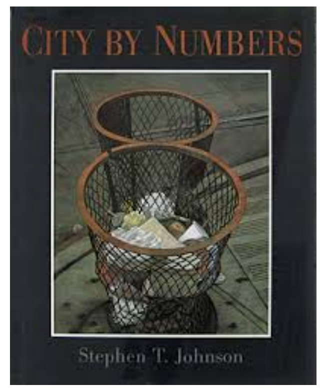
What in the World? Numbers in Nature
N. R. Day, K. Cyrus



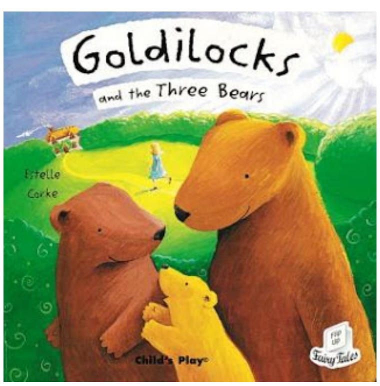
More
I.C. Springman, B. Lies



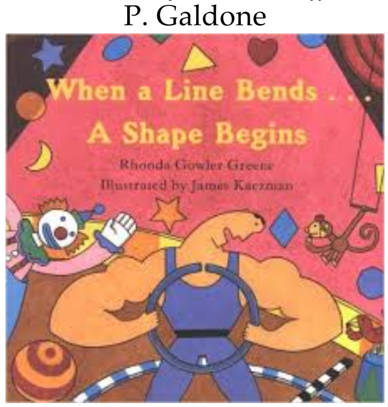
The Three Billy Goats Gruff
P. Galdone



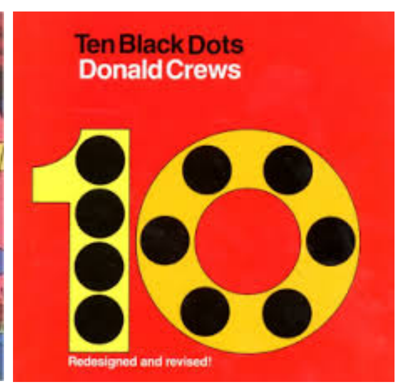
City by Numbers
S. T. Johnson



Goldilocks and the Three Bears
E. Corke



When a Line Bends… A Shape Begins
Rhonda Gowler Greene,
James Kaczman



Ten Black Dots
D. Crews

For more math-related books, see the blog post "Math Read Alouds: Connecting Math and Literacy" by Tracy Pickard and Cheryl Emrich (passionatelycuriousinkindergarten.blogspot.ca/2015/08/math-read-alouds-connecting-math-and.html).

Inquiries and Projects

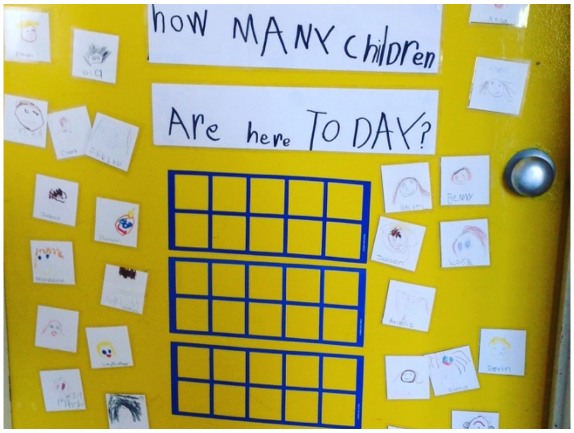
Math connections can often be found in inquiries and projects that are of interest to students. Finding these links allows for authentic and purposeful learning of mathematical concepts.

"Nonliving, Living, Don't Know"

Routines and Natural Occurrences

A lot of great math happens spontaneously! Taking advantage of opportunities to apply mathematical concepts to everyday experiences and events is a wonderful way to authentically support learning and to demonstrate the usage and importance of mathematics.

Educators can play an integral role by making meaningful connections between the mathematical strands, the real world and other disciplines and most importantly, between the intuitive informal mathematics that students have learned through their own experiences and the mathematics they are learning in school. (Ontario Ministry of Education, 2003, p. 14).



Other examples:

- Placing of lunch bags (one per table) for lunch
- Counting the water bottles on our two shelves so there are not more on one shelf than the other and create falls
- Noting the number of people at certain exploration areas
- Sorting materials during clean-up time
- Noting who is tall and short to figuring out who stands or kneels for reflection sharing so that all can see the work shared

Math in Play

Noticing and drawing attention to math-related content in children's play can greatly support learning by introducing appropriate mathematical language and concepts purposefully.

Knowledgeable educators recognize that although young children may have a beginning understanding of mathematical concepts they often lack the language to communicate their ideas. By modelling and fostering math talk throughout the day and across various subject areas, educators can provide the math language that allows students to articulate their ideas. (Ontario Literacy and Numeracy Secretariat, 2011, p. 4)

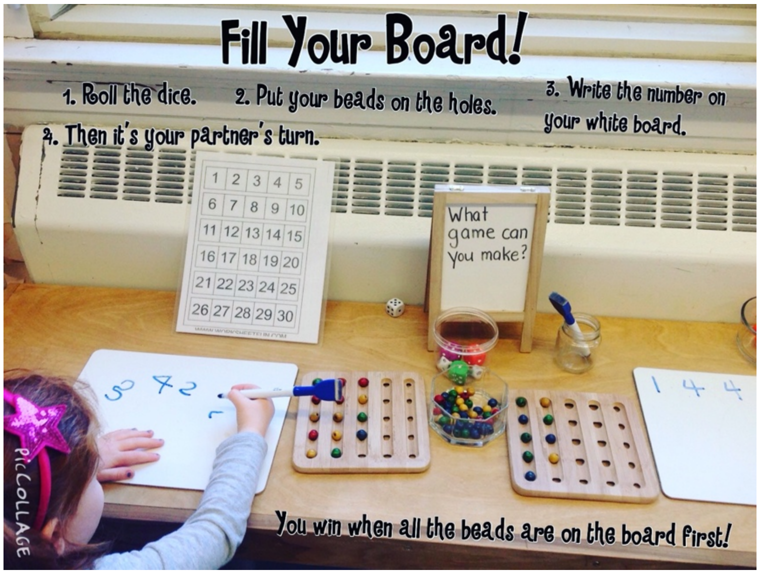
Games

Some benefits of games in the classroom:

- Playing games encourages strategic mathematical thinking as students find different strategies for solving problems and deepen their understanding of numbers.
- When played repeatedly, games support students' development and computational fluency.
- Games present opportunities for practice, often without the need for teachers to provide the problems.
- Games have the potential to allow students to develop familiarity with number system and engage in building a deeper understanding of operations. (Rutherford, 2015)

Here are some of the games that our students have played using dice, beads, and frames:

Fill Your Board!
Roll the dice. Put your beads on the holes. Write the number on your white board. You win when all the beads are on the board first!



Last Hole! (Student-created game)
"First you roll the dice. Then you pick a bead. Then put the bead on the hole you rolled. Then your partner rolls and does the same thing. You keep going until you roll the right number to win!"

Pop! (Student-created game)
"Roll the dice. What number you get put it on the board with beads. You keep playing until you cover all the holes. If you get 6 on the dice then you have to put all your beads back."

For more information about using games in the teaching and learning of mathematics in the early years, see the following articles and websites:

Cutler, K. M., Gilkerson, D., Parrott, S., & Bowne, M. T. (2003). Developing math games based on children's literature. *Young Children*, *58*(1), 22-27. Available at https://www.naeyc.org/files/yc/file/200301/MathGames.pdf

NRICH. (n.d.). Playing with dice. Available at http://nrich.maths.org/8380

NRICH. (n.d.). Teacher KS1 playing with dice collection. Available at http://nrich.maths.org/8371

Ramani, G. B., & Eason, S. H. (2015). It all adds up: Learning early math through play and games. *Phi Delta Kappan, 96 (8)*, 27-32. Available at http://www.kappancommoncore.org/it-all-adds-up-learning-early-math-through-play-and-games/

Rutherford, K. (2015, April 27). Why play math games? Available at http://www.nctm.org/publications/teaching-children-mathematics/blog/why-play-math-games__/

Way, J. (2011, February). Learning mathematics through games: Series 1. Why games? Available at https://nrich.maths.org/2489

Way, J. (2011, February). Learning mathematics through games: Series 1. Types of games. Available at http://nrich.maths.org/2491

Why Teach Math Holistically?

In our view, a holistic approach to learning mathematics, which we offer by giving students many opportunities to explore and investigate as co-teachers and co-learners, is very rewarding for the children, and not only in the early years. Exposure to a variety of experiences and activities allows children to apply, practice, and extend their knowledge in authentic and hands-on ways.

References

Copley, J. V., Jones, C., & Dighe, J. (2007). *Mathematics: The creative curriculum approach*. Washington, DC: Teaching Strategies.

Cutler, K. M., Gilkerson, D., Parrott, S., & Bowne, M. T. (2003). Developing math games based on children's literature. *Young Children*, *58*(1), 22-27.

McLennan, D. P. (2014, October/November). Making math meaningful for young children. *Teaching Young Children*, *8*(1), 20-22.

Ontario Literacy and Numeracy Secretariat. (2011, September). Maximizing student mathematical learning in the early years. *Capacity building series: Special edition #22.* Ontario Ministry of Education.

Ontario Ministry of Education. (2003). *Early math strategy: The report of the Expert Panel on Early Math in Ontario*. Toronto: Queen's Printer for Ontario.

Rutherford, K. (2015, April 27). Why play math games? [Web log post]. Retrieved from http://www.nctm.org/publications/teaching-children-mathematics/blog/why-play-math-games_/

Anamaria Ralph is a Kindergarten teacher for the Toronto District School Board. She teaches at Maurice Cody Public School in Toronto, Ontario. She has taught Kindergarten for nine years and still regards each year as an exciting adventure where many wonders, explorations, and investigations take place. She is passionate about inquiry and play-based learning, and is greatly inspired by the Reggio approach to learning. She shares her students' learning with families and other educators on her classroom blog, www.wondersinkindergarten.blogspot.ca, and can also be reached on Twitter at @anamariaralph.

Intersections

In this monthly column, you'll find information about upcoming math (education)-related workshops, conferences, and other events that will take place in Saskatchewan and beyond. If travel is not an option at this time or if you prefer learning from the comfort of your own home, see the Online workshops and Continuous learning online sections. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description.

Within Saskatchewan

Conferences

Saskatchewan Understands Math (SUM) Conference
November 4th – 5th, Saskatoon, SK
Presented by the SMTS

Our own annual conference! The Saskatchewan Understands Math (SUM) conference is for math educators teaching in K-12 who are interested in curriculum, incorporating technology, number sense, and problem solving. Join us for two days packed with learning opportunities, featuring keynote speakers Max Ray-Riek of the Math Forum at NCTM and Grace Kelemanik of the Boston Teacher Residency Program. Registration includes lunch on Friday and a two-year SMTS membership. See poster on page 3, and head to our website for more information and to register.

Interested in presenting? The planning committee is seeking 60-minute presentations on topics related to the teaching and learning of mathematics. Presenters will be provided with one free conference registration per session. Submit your proposal on our website by September 16, 2016.

Workshops

Structures for Differentiating Middle Years Mathematics
September 26th, Regina, SK
$110 (early bird), $150 (standard)
Presented by the Saskatchewan Professional Development Unit

We know that assessing where students are at in mathematics is essential, but what do we do when we know what they don't know? What do we do when they DO know? Student understanding does not change unless there is an instructional response to an assessment. This workshop will introduce an Assess-Respond- Instruct Cycle in mathematics, as well as responsive stations as a classroom structure to meet individual student needs, without having to create a completely individualized mathematics program in your classroom.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/structures-differentiating-middle-years-2

Number Talks and Beyond: Building Math Communities Through Classroom Conversation
November 16th, Saskatoon, SK
$110 (early bird), $150 (standard)
Presented by the Saskatchewan Professional Development Unit

Classroom discussion is a powerful tool for supporting student communication, sense-making and mathematical understanding. Curating productive math talk communities requires teachers to plan for and recognize opportunities in the live action of teaching. Come experience a variety of classroom numeracy routines including number talks, counting circles, quick images and more. Take math conversations to the next level by strengthening your skills as a facilitator of classroom discourse and student thinking.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/number-talks-and-beyond-building

Beyond Saskatchewan

MCATA Fall Conference 2016: Opening Your Mathematical Mind
October 21-22, Canmore, AB
Presented by the Mathematics Council of the Alberta Teachers' Association

Come join the Mathematics Council of the Alberta Teachers' Association in celebrating their annual fall conference "Opening Your Mathematical Mind" at the Coast Canmore Hotel & Conference Centre, 511 Bow Valley Trail, Canmore, Alberta. Featuring Keynote Speakers Dr. Peter Liljedahl of Simon Fraser University and Dr. Ilana Horn of Vanderbilt University. See https://event-wizard.com/OpeningYourMathmind/0/welcome/

55th Northwest Mathematics Conference
October 21-23, Yakima, WA

The Northwest Mathematics Conference is a collaborative conference held annually, alternating between Washington, Oregon, and British Columbia. The target audience of preK-16 math educators includes pre-service, active, and retired elementary, middle, and high school teachers, community college and university instructors, math coaches, staff development specialists, and special needs and ELL math teachers.

Approximately 1,000 participants will gather in Yakima for this year's two-day conference, which will kick off with a Maker's Fair and is centered around the theme "What is Next in Mathematics? WIN with Math." This year's keynote speaker is Michael Stevens (Vsauce1); featured speakers are Steve Leinwald, Ruth Parker, and Sandy Atkins. Five strands will be highlighted across the event: Early Numeracy – Setting the Foundation for the Future; the CCSS Standards for Mathematical Practice – Engaging Students in Learning; Post-Secondary Education – Preparing for Tomorrow; STEAM – Driving Innovation in Learning, and Assessment – Deepening Understanding.

See http://www.northwestmathconference2016.org/

Online Workshops

Math Daily 3
July 31th–August 22nd or *August 28–September 24*
Presented by the Daily CAFÉ

Learn how to help your students achieve mathematics mastery through the Math Daily 3 structure, which comprises Math by Myself, Math with Someone, and Math Writing. Allison Behne covers the underlying brain research, teaching, and learning motivators; classroom design; how to create focused lessons that develop student independence; organizing student data; and differentiated math instruction. Daily CAFE online seminars combine guided instruction with additional resources you explore on your own, and are perfect for those who prefer short bursts of information combined with independent learning.

The seminar includes:
- online access to videos, articles, and downloadable materials
- access to an exclusive online discussion board with colleagues
- a certificate of attendance for 15 contact hours

See https://www.thedailycafe.com/workshops/10000

Continuous Learning Online

Education Week Math Webinars
Presented by Education Week

Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling and Differentiation.

Past webinars: http://www.edweek.org/ew/webinars/math-webinars.html
Upcoming webinars:
http://www.edweek.org/ew/marketplace/webinars/webinars.html

Call for Contributions

Did you just deliver a great lesson? Or maybe it didn't go as planned, but you learned something new about the complexities of teaching and learning mathematics. Maybe you just read a book or attended a workshop that gave you great ideas for presenting a topic your students have always found difficult, or that changed your perspective about some aspect of teaching. **Why not share your ideas with other teachers in the province – and beyond?**

The Variable is looking for a wide variety of contributions from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants and coordinators, teacher educators, researchers, and students of all ages. Consider sharing a favorite lesson plan, a reflection, an essay, a book review, or any other article or other work of interest to mathematics teachers in Saskatchewan. If accepted for publication, your piece will be shared, as part of this periodical, with a wide audience of mathematics teachers, consultants, and researchers across the province, as well as posted on our website.

We are also looking for student contributions, whether in the form of artwork, stories, poems, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom and their school, and for teachers to recognize their students' efforts during their journey of learning mathematics.

All work will be published under a Creative Commons license. If you are interested in contributing your own or (with permission) your students' work, please contact us at thevariable@smts.ca.

We look forward to hearing from you!