The Variable

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Saskatchewan Mathematics Teachers' Society

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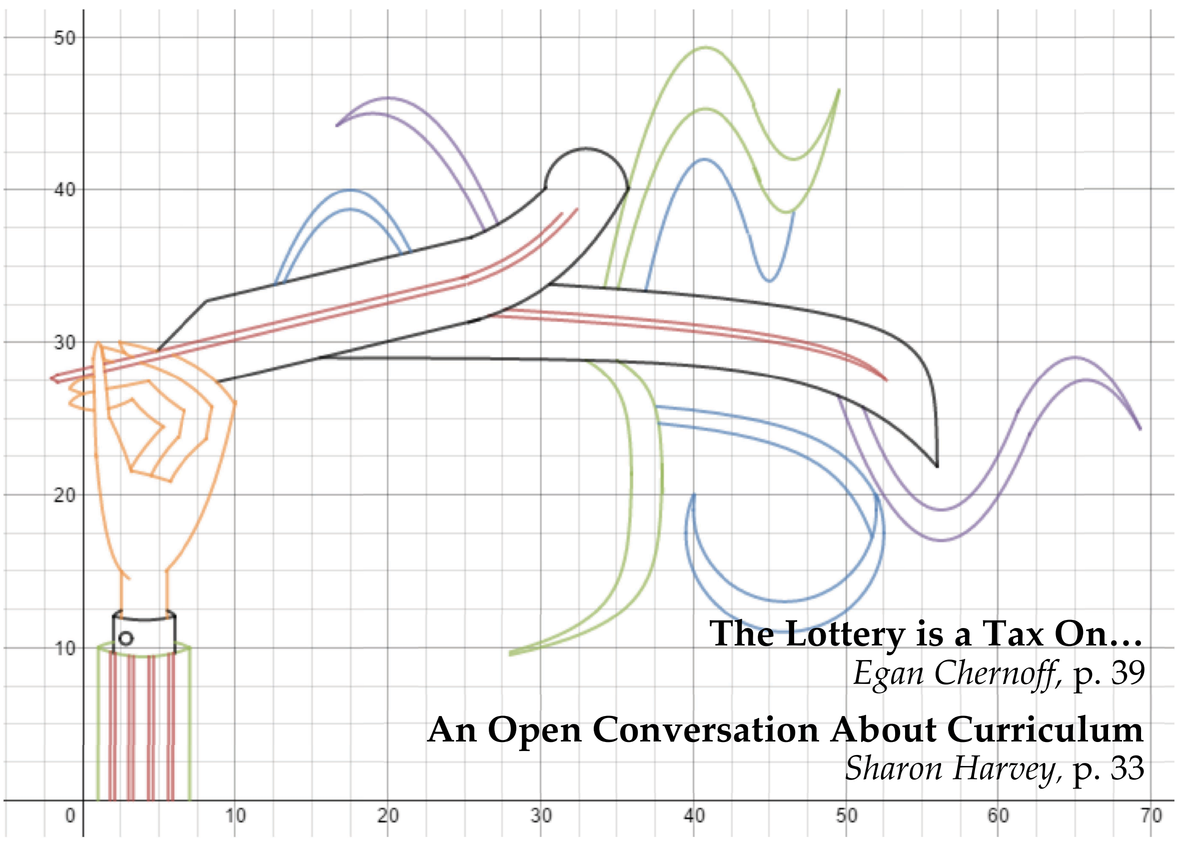

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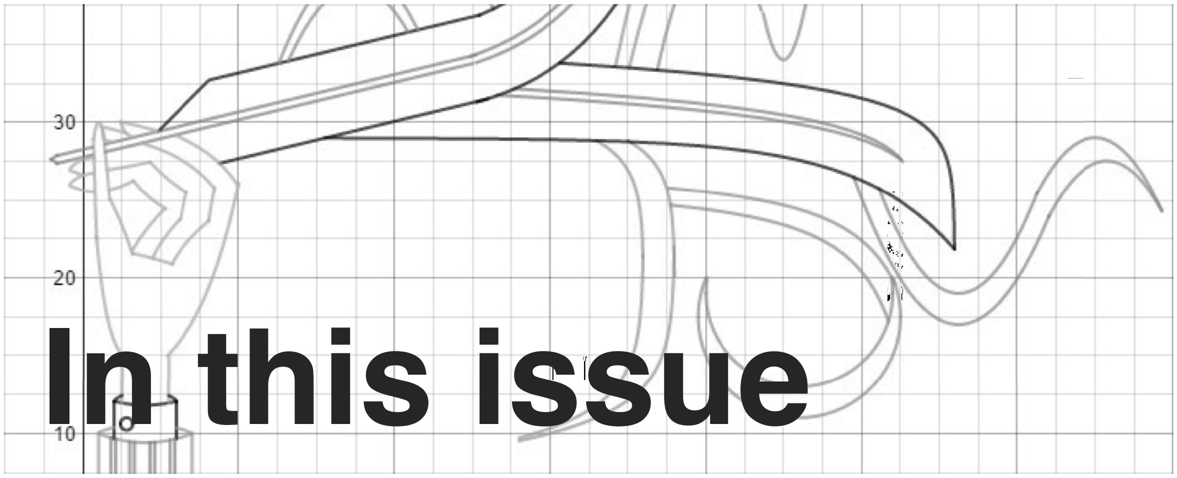

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Cover Image

This month's cover image was created by a Grade 12 student using the online graphing calculator Desmos as part of a term project on function transformations. For more information about the project, see the article "Desmos Art" by Nat Banting on page 25.

SMTS Executive (2016-2018)

Notice to Readers

The views expressed in *The Variable* are those of the author(s), and not necessarily those of the Saskatchewan Mathematics Teachers' Society. In the publishing of any works herein, the editorial board has sought confirmation from all authors that personal quotes and examples have been included with the permission of those to whom they are attributable.

Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants, teacher educators, researchers, and students of all ages, although we encourage Saskatchewan teachers of mathematics as our main contributors. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Articles should be limited to 3000 words or less; authors should also include a photo and a short biographical statement of 75 words or less. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Call for Contributions

Did you just teach a great lesson? Or maybe it didn't go as planned, but you learned something new about the complexities of teaching and learning mathematics. Maybe you just read a book or attended a workshop that gave you great ideas for presenting a topic your students have always found difficult, or that changed your perspective about some aspect of teaching. **Why not share your ideas with other teachers in the province—and beyond?**

The Variable is looking for contributions from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants and coordinators, teacher educators, and researchers. Consider sharing a favorite lesson, a reflection, an essay, a book review, or any other work of interest to mathematics teachers in Saskatchewan. If accepted for publication, your piece will be shared with a wide audience of mathematics educators in Saskatchewan and beyond.

We are also looking for student contributions in the form of artwork, stories, poems, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom and their school, and for teachers to recognize their students' efforts during their journey of learning mathematics.

All work is published under a Creative Commons license. If you are interested in contributing or have any questions, please contact us at thevariable@smts.ca.

We look forward to hearing from you!


Saskatchewan Mathematics Teachers' Society
SELU
Leading Together
The SMTS, SELU and SPDU presents ...
#SUM2017
Who: K-12 teachers, coaches, consultants, coordinators, superintendents and directors
Where: TCU Place, Saskatoon
When: October 23-24, 2017
Cost: $315 (early registration) | $375 (regular)
Keynote Presenters
Lisa Lunney Borden, St. Francis Xavier University
Steve Leinwand, American Institutes for Research
Featured Presenters
Kristin Gray, Cape Henlopen School District, Illustrative Mathematics
Glenn Waddell, The University of Reno, Desmos Fellow
Egan Chernoff, University of Saskatchewan
Gale Russell, University of Regina
NCTM
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Saskatchewan Understands Mathematics Conference 2017

Keynotes

Steve Leinwand

Practical Suggestions for Building a Powerful and Professional 2017-2018 To-do List

This fast-paced and example-laden pep-talk will discuss and model a set of instructional shifts that NCTM's Mathematical Teaching Practices and the quest for more effective instruction require us to consider in order to enhance our teaching and our students' learning.

Featured Session 1: Proving the Leadership Necessary for Making Mathematics Work for All Students (Part 1)
We know that effective programs of K-12 mathematics require informed and effective leadership. This part one of a two-part series of workshops will focus on specific understandings that every mathematics leader needs to have about effective mathematics programs, with a focus on high quality instruction, to be in a position to advocate for and support such programs.

Featured Session 2: Proving the Leadership Necessary for Making Mathematics Work for All Students (Part 2*)
We know that effective programs of K-12 mathematics require informed and effective leadership. This part two of a two-part series of workshops will focus on specific strategies and initiatives that every mathematics leader needs to establish, nurture and monitor to ensure that the effective mathematics programs discussed during part one are available to all students in every school. We'll take particular look at a range of collaborative structures that reduce professional isolation and support professional growth.

*attendance at Part 1 is not required to participate in Part 2

Lisa Lunney Borden

The Role of Mathematics Education in Reconciliation

The 2015 TRC final report that includes calls to action in response to the horrors of residential schools for Aboriginal Canadians that are focused on establishing a renewed relationships between non-Aboriginal and Aboriginal Canadians to "restore what must be restored, repair what must be repaired, and return what must be returned" (2015, p. 6). The TRC names the education system as having an essential role in repairing the damages caused by residential schools. Lisa will reflect on her 22-year career as a teacher and researcher working in Indigenous communities, primarily Mi'kmaw communities, to explore the role of mathematics education in reconciliation. She will share stories of hope and healing that have emerged through the Show Me Your Math program, inquiry projects, outreach programs, and teacher professional learning that give insights into how mathematics can aid in reconciliation.

Featured Session 1: Our Ways of Knowing: Teaching Math with Verbs and Space
Lisa will share a model for considering ways in which Indigenous languages, community values, ways of knowing, and cultural connections can impact mathematics learning for Indigenous learners. Participants will go more deeply into the pedagogical implications of this model that are linked to the ways of knowing that emerge from an understanding of the structure of Indigenous languages. We will engage in tasks that highlight the value of verbifying and spatializing mathematics teaching and learning. Examples will be drawn from Kindergarten to Grade 12 to highlight how these approaches span all levels.

Featured Session 2: My Elders were Mathematicians Too: The Value of Culturally-based Inquiry
Lisa will share the story of Show Me Your Math, a program that invites Indigenous students in Atlantic Canada to explore the mathematics that in inherent in community ways of knowing, being, and doing. She will share the history of this program, how it has changed over time to focus more on inquiry, and how it might be developed in other regions. We will explore examples of projects that have been completed, examine the benefits of these projects and discuss how such projects help to restore, reclaim, and return community knowledge that has been eroded by colonialism.

For more information on keynote and featured speakers, please visit:
http://smts.ca/sum-conference/sum-keynote-presenters

Message from the President

Another school year done! A huge congratulations and thank you to every single one of our members for the contribution they make to the students of Saskatchewan (and elsewhere), whatever your role. This year was a hard year politically on what feels like every level of politics, and in many ways it makes this year's ~~unpaid days off~~ summer holidays all the more deserved, should you have them. Regardless of your summer status, we hope that you have plans for whatever rejuvenates and restores you.

For many, summer can also offer little pockets of opportunity to do some planning and preparing for next year. What a great time to browse the archives of *The Variable* or to hop over to Twitter to engage with some of the SMTS executive or other *Variable* authors as you think about how you might incorporate some new ideas into your classroom next year.

Another integral part of planning is reflecting on the year. The SMTS executive had an opportunity to come together in May and reflect on our year, and what a year it was! We set some very lofty short- and long-term goals when we participated in strategic planning with the Saskatchewan Professional Development Unit just a little over a year ago, and have needed to modify along the way. However, overall, it sure was satisfying to see that as an organization, we are exactly where we'd hoped to be at this time this year.

We hope you're loving hearing from us as much as we enjoy communicating with you. While we did adjust our original publication schedule to every two months instead from monthly, I can't congratulate the Communications team enough for all their hard work on bringing the SMTS blog and *The Variable* to life. I'd be remiss if I didn't remind you that a quiet summer evening on the deck is a great time to type up that thing that went really great for you in math class to share with other teachers through *The Variable*. If this is a new process for you, our editing team is happy to offer support and to answer any questions you may have.

Lastly, I'd like to remind you that now is a great time to consider getting more involved with the SMTS. It's an election year for our executive, and our board (very sadly) will have some vacancies. Our annual general meeting will be held on Monday, October 23 at SUM Conference. In case you need another reason to attend, this year's SUM Conference is a very special partnership between the SMTS, the Saskatchewan Educational Leadership Unit (SELU), and the Saskatchewan Professional Development Unit (SPDU). The Ministry of Education and the Provincial Leadership Team have also joined us as advisory partners—so grab a teacher partner, your in-school administrator, and your superintendent and join us for two amazing days of learning.

With this, I'm going to wish you a most restful summer. See you back here in September!

Michelle Naidu

Problems to Ponder

Welcome to the July/August edition of Problems to Ponder! This collection of problems has been curated by Michael Pruner, president of the British Columbia Association of Mathematics Teachers (BCAMT). The tasks are released on a weekly basis through the BCAMT listserv, and are also shared via Twitter (@BCAMT) and on the BCAMT website.

Have an interesting solution? Send it to thevariable@smts.ca for publication in a future issue of The Variable.

British Columbia Association of Mathematics Teachers

I am calling these problems 'competency tasks' because they seem to fit quite nicely with the curricular competencies in the British Columbia revised curriculum. They are non-content based, so that all students should be able to get started and investigate by drawing pictures, making guesses, or asking questions. When possible, extensions will be provided so that you can keep your students in flow during the activity. Although they may not fit under a specific topic for your course, the richness of the mathematics comes out when students explain their thinking or show creativity in their solution strategies.

I think it would be fun and more valuable for everyone if we shared our experiences with the tasks. Take pictures of students' work and share how the tasks worked with your class through the BCAMT listserv so that others may learn from your experiences.

I hope you and your class have fun with these tasks.

Michael Pruner

Primary Tasks (Kindergarten-Intermediate)

Number Path[1]

You are on a number path made up of squares of numbers starting at 1 and continuing as far as you wish…

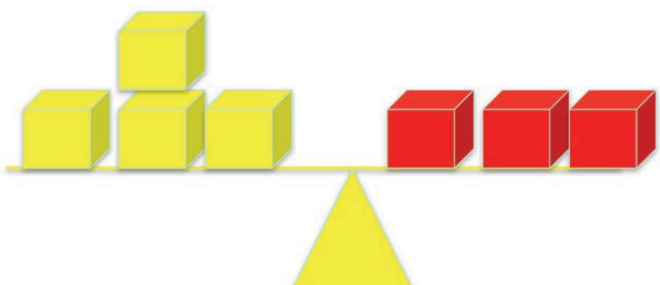
[1] Small, M. (2017). Mathematical problem solving in all strands [PowerPoint slides]. Retrieved from http://www.onetwoinfinity.ca/wp-content/uploads/2017/03/WinnipegMarch.pdf

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**

You move *some* steps forward. Then, you move *some* steps back. You repeat both moves. You land at 9. How many steps did you take each way?

Sharing Cookies[2]
Charlie, Susan, and Amber get to share six cookies. However, Susan's mother has told her that she is only allowed to have one cookie. How do you share the cookies?

How Many?[1]
How many might be in each yellow box? How many in each red box?



Intermediate and Secondary Tasks (Intermediate-Grade 12)

25 Coins[3]
Twenty-five coins are arranged in a 5 by 5 array. A fly lands on one, and tries to hop on to every coin exactly once, at each stage moving only to the adjacent coin in the same row or column. Is this possible?

Extensions: Can you explain why some starting locations are not possible? What about 3D? What about rectangles?

[2] Winter 2010 problem set. (2010). *Vector*, *51*(1), 40-43. Retrieved from http://www.bcamt.ca/wp-content/uploads/vector/511-Winter-2010.pdf

[3] Mason, J., Burton, L, & Stacey, K. (1985). *Thinking mathematically*. Essex, England: Prentice Hall.

Split 25[4]

Take the number 25, and break it up into as many pieces as you want. For example,

25 = 10 + 10 + 5
25 = 2 + 23
25 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 9 + 9

What is the biggest product you can make if you multiply those pieces together? Will your strategy work for any number?

Consecutive Sums[3]

Some numbers can be expressed as the sum of two or more consecutive positive integers. For example,

9 = 2 + 3 + 4
11 = 5 + 6
18 = 3 + 4 + 5 + 6

Which numbers have this property?

Extensions: In how many ways can a number *n* be written as the sum of two or more positive integers?

Michael Pruner is the current president of the British Columbia Association of Mathematics Teachers and a full-time mathematics teacher at Windsor Secondary School in North Vancouver. He teaches using the Thinking Classroom model where students work collaboratively on tasks to develop both their mathematical competencies and their understanding of the course content.

[4] Kelly, J., & Yu, X. (n.d.). *Play with your math.* Retrieved from http://www.playwithyourmath.com/

Spotlight on the Profession

In conversation with Steve Leinwand

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Steve Leinwand, who we look forward to welcoming this fall as a SUM Conference 2017 keynote presenter.

Steve Leinwand is a Principal Research Analyst at the American Institutes for Research (AIR) and has over 35 years of leadership positions in mathematics education. He currently serves as mathematics expert on a wide range of AIR projects that focus on high quality mathematics instruction, turning around underperforming schools, evaluating programs, developing assessments and providing technical assistance. Leinwand has spoken and written about effectively implementing the Common Core State Standards in Mathematics, differentiated learning, and "What Every School Leader Needs to Know about Making Math Work for All Students." In addition, Leinwand has overseen the development and quality review of multiple-choice and constructed response items for AIR's contracts with diverse states.

Before joining AIR in 2002, Leinwand spent 22 years as Mathematics Consultant with the Connecticut Department of Education, has served on the National Council of Teachers of Mathematics' Board of Directors, and has been President of the National Council of Supervisors of Mathematics. Steve is also an author of several mathematics textbooks and has written numerous articles. His books, Sensible Mathematics: A Guide for School Leaders in the Era of Common Core State Standards *and* Accessible Mathematics: 10 Instructional Shifts That Raise Student Achievement, *were published by Heinemann in 2012 and 2009, respectively. In addition, Leinwand was the awardee of the 2015 National Council of Supervisors of Mathematics Glenn Gilbert/Ross Taylor National Leadership Award for outstanding contributions to mathematics education.*

First things first, thank you for taking the time for this conversation!

With over 35 years of leadership positions in mathematics education that span consulting, evaluation, program development, research, and more, you have surely observed many changes in curriculum, pedagogy, assessment, and philosophy in the area of mathematics teaching and learning at the primary and secondary levels.

In your view, what are we doing better today in the area of mathematics education, in comparison to 35 years ago?

I am very optimistic and believe that nearly every aspect of the teaching and learning of mathematics is better than it was 35 years ago. In terms of curriculum, particularly in Grades K-8, we have moved from fragmented, repetitive and overwhelmingly skill-based mathematics to fewer and clearer standards, and more teachable mathematics based on coherent progressions that significantly eliminate duplication. In terms of instruction, we have moved from worksheet and practice-driven teaching-by-telling to more problem-driven, activity-driven learning-by-doing. And every year reveals important shifts in assessment away from primarily multiple-choice assessment of skills to far more open-ended, constructed-response assessments of a balance of skills, concepts, and applications. It's not just my optimism or seeing things through rose-colored glasses: rather, these changes are reflected in consistently higher scores on a range of reliable measures of student achievement. Back in 1990, only a dismal 13 percent of US fourth graders were deemed proficient or above on the National Assessment of Educational Progress. In 2013, this had grown significantly to 42 percent! This is still far short of where we need to be, but it is certainly clear evidence of real change and improvement.

And where do you see there being greatest room for improvement?

> "No longer just frills or add-ons, collaborative structures and coaching must be seen as essential components of a system where our goal of high levels of mathematics for all students can be met."

There is no question that, given the improvements in the curriculum and our assessments, and the increasing availability of high-quality instructional materials, our primary challenge is bringing high-quality instructional practice to scale in every school and classroom. This requires far more professional sharing and far more professional transparency. That is, my sense is that the greatest needs for change are collaborative structures and coaching – two elements of our professional culture that are undervalued at our peril. Show me a grade level or a department that meets regularly to discuss and share plans and strategies, that looks at assessment data, that observes and critiques each other in real time or via video, and I'll show you a cadre of teachers better motivated and empowered to be more effective. Then show me a school where there is a half-time to full-time coach who lives in classrooms and grade level meetings, who earns the respect of the teachers being coached through co-teaching, constructive feedback, and the provision of ideas and resources, and again, I'll show you a school or department positioned to best serve its students. No longer just frills or add-ons, these collaborative structures and coaching must be seen by all as essential components of a system where our singular goal of high levels of mathematics for all students can actually be met.

The National Council of Teachers of Mathematics' (NCTM) Curriculum and Evaluation Standards for School Mathematics *(1989) and, then, the* Principles and Standards for School Mathematics *(NCTM, 2000) have been major drivers of changes in the teaching and learning of mathematics in the past, and are still influential today. As Alan Schoenfeld (2004, p. 266) wrote, "none of the authors or others involved in the production of the [1989]* Standards *had any idea of what the ultimate magnitude of the response to their document would be," and yet they spurred "a highly creative design process during the following decade" (p. 268) as a "'standards movement' took the nation by storm" (p. 269), setting in motion changes to curriculum, pedagogy, and materials across the country—and indeed, the continent, influencing as they did the Western and Nothern Canadian Protocol (WNCP).*

What do you see as being the greatest drivers of change in mathematics education today, and in the coming decades?

> "I see four powerful levers or drivers of change in mathematics education: all, resources, tests and technology."

I see four powerful levers or drivers of change in mathematics education: all, resources, tests and technology. First, for years and years, mathematics was systematically designed to sort students out. The good got better, the average stayed average and the weak got weaker, and we just assumed that that was the way it was. Finally, changes in the workplace and technology have helped to convince us that today, math must work for *all.* This push from some to all, and its implications for curriculum and instruction is a prime driver. Second, the internet and the amazing range of free, or essentially free, resources – many of very high quality – have also resulted in significant change and improvement. I have been overjoyed by the use of three-act lessons [first introduced by California teacher Dan Meyer –*Ed.*], Desmos activities, great tasks, virtual manipulatives, and much, much more to improve student access to and engagement with mathematics. Third, our tests seem to get better every year and also drive change and improvement as constructed-response replaces multiple-choice and computer-adaptive replaces fixed-form paper-and-pencil. Finally, when any 15-year-old can use his or her smartphone's calculator and a free download of Desmos, we have changed the game from expensive graphing calculators and web-based applications requiring internet access to readily available tools that, when used appropriately, change what math is important and how best to teach this math for understanding. Coupled with the power of display that emerges from interactive white boards and document cameras, technology is the fourth critical driver of powerful changes in teaching and learning.

In Leinwand (2009, par. 1), you suggest that "a strong K-12 mathematics program [is] at the heart of America's long-term economic viability," given that long-term economic security and social well-being are linked to sustained innovation and workplace productivity, which in turn rely on high-quality education in the areas of literacy, mathematics, and science.

And yet, with the proliferation of technology, it is clear that the world does not need human calculators solving artificial problems that can be computed in milliseconds by a pocket calculator or smartphone.

If this is the case, what kinds of skills and habits of mind can students develop during their mathematics education that are relevant and necessary in the modern world, and how has mathematics curriculum and pedagogy changed—or should change—to emphasize these skills?

My touchstone for what is non-negotiable for preparation for the 21st century world of work and effective citizenship is the first four Standards for Mathematical Practice delineated in the Common Core. That is, when *all* students can persevere and solve problems, reason quantitatively and mathematically, model with mathematics and most importantly, construct viable arguments and critique the reasoning of others, we have truly prepared our students for an ever-changing and increasingly complex world. Every mathematics lesson and every mathematics assessment must be planned and implemented with these four practices in mind. If there are no problems to solve in a lesson, if students are not asked "why?" or "can you convince us?" to construct an argument and demonstrate reasoning, and if the lesson is essentially about how to find an answer without understanding, then we know we are *not* preparing our students with the skills and understandings and practices they really need.

You have spent some time studying high-performing education systems around the world (e.g., Singapore; Ginsburg, Leinwand, Anstrom, & Pollock, 2005) in an effort to unearth high-impact practices in mathematics education.

In a nutshell, what can North America learn from the mathematics education programs in countries such as Singapore?

What we learned in our study of Singapore's impressive mathematics program was very simple: First, there is high quality to each of the components of their program, and second, there was strong alignment among each of these components. That is, Singapore Math is not just a textbook, a philosophy, or a system of assessment. Rather, K-6 mathematics in Singapore is based on a clear and coherent set of standards, strong instructional materials, and effective teaching supported by impactful professional development, and is held together with high quality assessments, where each of these components is carefully and closely aligned with the others. Our study of Singapore math helped to influence the Common Core Standards movement in the US with a much more coherent set of standards and higher-quality, aligned assessments. The market has since provided increasingly aligned curricular materials.

And which aspects of these programs do not *transfer well—due to cultural differences, or otherwise—to mathematics education programs in North America?*

I think that all of the components of Singapore Math transfer well to North America, with one exception. One often hears from mathematics educators in Singapore: "What's your problem in North America? All that we in Singapore have done is take your research and implemented it. There's no magic. Just do what your NCTM, your Common Core standards, and your research tell you to do." That is, when it comes to content standards, curriculum materials, instruction, assessments and their alignment, these is no reason at all that these elements cannot be transferred to Canada and the US.

> "Where there remain gigantic gaps between Singapore and North America is in the domain of teacher recruitment, training, induction, and support."

However, where there remain gigantic gaps between Singapore and North America is in the domain of teacher recruitment, training, induction, and support. In Singapore, teacher trainees are paid throughout two years of intensive training in both mathematics and the teaching of mathematics (granted, Singapore is a city-state about the size of Chicago). Singapore teachers then serve as interns during an intensive year of induction, only observing their colleagues and co-teaching. Even during their second year in schools, Singapore teachers are very closely supervised and mentored, finally becoming independent practitioners in their third year, during which they now mentor first year teachers. Obviously, there is a lot that we in North America have to do to begin to replicate practices like this.

You have often discussed the importance of coherence and alignment in mathematics curricula between goals, materials, instruction, assessment, and so on (e.g., Leinwand, 2009, 2012). As you write in Leinwand (2012, par. 7), in the context of presenting strategies to improve an underperforming mathematics education program: "You need a coherent and aligned curriculum that includes a set of grade level content expectations, appropriate print and electronic instructional materials, with a pacing guide that links the content standards, the materials and the calendar."

You have also, however, applauded the growth of online resources available for teachers, such as Dan Meyer's three-act tasks, Andrew Stadel's Estimation 180 lessons, and more (Editorial, 2015). Many of these resources have been developed by individual classroom teachers or consultants, rather than teams of curriculum developers.

Is there a tension here—that is, a tension between a desire for coherence and the (ever-increasing) diversity of readily-available resources? Is coherence possible in the digital age, where teachers pull together content from a variety of resources (high-quality or otherwise) during their lesson planning?

No, I do not see a tension here and yes, coherence is entirely possible in this digital age. First, we agree on a coherent, balanced, and teachable set of standards for each grade or course. This is the guiding skeleton for selecting tasks, questions and assessments. Such standards are increasingly available for Grades K-8, but 9-12 continues to be a bloated hodgepodge of increasingly obsolete skills where we are asked to add statistics and modeling, but given no guidance on what to remove and no more time with which to teach. Since no one resource is likely to align perfectly with the selected standards, it is entirely appropriate to carefully pick and choose or cut and paste from the growing wealth of on-line resources to arrive at the right tasks, the right questions, and the right assessments that together support the successful meeting of the standards.

"Since no one resource is likely to align perfectly with the selected standards, it is entirely appropriate to carefully pick and choose from the growing wealth of on-line resources."

Lastly, our readers are likely aware that you will in Saskatoon this November to present as a keynote speaker at our very own Saskatchewan Understands Math (SUM) Conference. (We can't wait!) We don't want to spoil the surprise, but could you give our readers some insight into what you will be discussing during your sessions?

I have been blessed to have been asked to give between five and ten such keynote talks each year. Accordingly, since I tend to get bored faster than anyone in my audience, I have tried to create a new keynote or major talk each year that I massage and revise over several months and then retire from my repertoire. This year's theme is designing lessons that incorporate the eight Mathematics Teaching Practices presented in NCTM's Principles to Actions. So my SUM talk will explore and model a lesson development process that includes goals, tasks, representations, discourse, questions, fluency, struggle, and evidence, all rolled up into an accessible process that supports effective teaching.

Thank you for taking the time for this conversation. We look forward to continuing the discussion at SUM Conference 2017!

Ilona Vashchyshyn

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A Preschool Investigation: The Skyscraper Project [5]

Kelly K. McCormick and Guinevere Twitchell

Young children are powerful mathematicians who are capable of demonstrating this power through their actions in both play and structured learning (Perry & Dockett, 2008). As educators, we must create the opportunities for children to use mathematics to make sense of the world around them. Young children thrive in classrooms that allow them to explore and discover their environment and interests and also support them in this learning. Because children learn best when they are interested and excited, early-childhood educators should offer children play-based, integrated mathematical experiences (NRC, 2009). In this article, we describe a meaningful project-based learning experience that intrinsically invites problem solving and mathematical thinking in a preschool classroom. Project-based learning is a learner-driven approach to teaching in which children investigate significant, real-world ideas or problems. Early-childhood, project-based learning allows children to learn through exploring their world of play and further investigate their curiosities and interests.

> "Young children thrive in classrooms that allow them to explore and discover their environment and interests."

The idea for the Skyscraper project emerged from observing children's spontaneous play and exploration. Our preschoolers love to build; they build with large blocks, small blocks, wooden blocks, unit blocks, magnetic building shapes, and Duplos®—sometimes all together! Block building supports children's learning of shape and shape composition and helps develop foundational spatial skills (Clements & Sarama, 2009). Initially we observed that after a brief period of building out, children's interest usually turned to building upward. However, their tall structures would fall because of unstable foundations. As we observed how building skyscrapers with blocks requires eye-hand coordination, spatial awareness, size comparison, concentration, and problem solving, we realized that the study of building-block skyscrapers provides a meaningful learning experience that naturally generates problem solving and mathematical thinking. We chose the project of investigating and constructing skyscrapers to support our students' mathematical reasoning processes, such as spatial and quantitative reasoning, because these are foundational to young children's mathematical development (Mulligan & Mitchelmore, 2013). Subsequently, we asked the children what they knew about skyscrapers. Some of their answers follow:

- "They're tall."
- "They have square windows."
- "It has a big round thing with a circle on top."
- "They scrape the sky. You can see the whole city. They move a little."

With these responses in mind, our Skyscraper investigation was born. We began by brainstorming activities that would undergird our integrated and emerging curriculum. As we considered our learners, we also considered the mathematics we wanted to support during the investigation (see Table 1). Constructing skyscrapers appeared to be a natural

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[5] Reprinted with permission from A Preschool Investigation: The Skyscraper Project, *Teaching Children Mathematics 23*(6), copyright 2017 by the National Council of the Teachers of Mathematics (NCTM).

way to support our students' numeric, geometric, and spatial reasoning while fostering their interests and recognizing and responding to the mathematics that emerged in their play. Early-childhood instruction should be rooted in play to provide the most developmentally appropriate approach and support children's growth in multiple domains (Wager, 2013).

TABLE 1

Content addressed throughout the project is matched with student activities.

Content and corresponding activities

Content area	Knowledge and skills	Activities
Number and number sense	Counting sequence Cardinality Estimation Compare numbers Recognize and write numbers	Constructing the cardboard city Construction books Building skyscrapers with various materials during free play Fill the skyscraper game
Measurement	Recognize, describe, and compare measurable attributes Measure with rulers Use standard and nonstandard units of measurement	Constructing the cardboard city Building skyscrapers with various materials during free play
Geometry	Recognize and name two- and three-dimensional shapes Create, compose, and decompose two- and three-dimensional shapes Draw two- and three-dimensional shapes Model shapes in the world Spatial sense Visualization Use spatial relational words	Constructions books Constructing the cities of skyscrapers Constructing two-dimensional skyscrapers with felt shapes (activity not mentioned in article) Constructing two- and three-dimensional shapes with straws and marshmallows Building skyscrapers with various materials during free play Quick Images of skyscrapers made of two-dimension shapes (activity not mentioned in article)
Data	Create bar graphs Use bar graphs to compare data	Constructing the cardboard city

We wanted to begin the investigation with a captivating experience that would obtain and hold the children's attention. Meaningful projects powerfully activate children's need-to-know with an engaging entry event (Lamar & Mergendoller, 2010). For our entry event, the children created their own large-scale cardboard skyscraper city. Research suggests that the preschool teachers should—

> provide materials, facilitate peer relationships, and time to build, and also incorporate planned, *systematic* block building into their curriculum. Children should have open exploratory play and solve semi-structured and well-structured problems, with intentional teaching provided for each. (Clements & Sarama, 2009, p. 151)

Constructing a cardboard city presented a perfect mix of excitement and play that was more structured.

Constructing a city of skyscrapers

We collected cardboard boxes and created a construction site, which evolved into the city of skyscrapers, by cordoning off an area with yellow tape wrapped around cones. We wanted to create a natural environment where the children had to communicate about the process of constructing skyscrapers, so each child worked collaboratively with a partner. The teams selected materials and designed and constructed their buildings on site. The structures were not originally taped together, so that students could dismantle and reconstruct them, an act that further supported their spatial thinking, and so that the parts could be moved easily for painting. Constructing, deconstructing, and then reconstructing the buildings with partners forced students to communicate about the spatial relationship between the boxes they had used. Working with partners created a natural situation in which the children practiced using spatial-relationship vocabulary, such as "*above, below, beside, in front of, behind,* and *next to*" (CCSSI, 2010, p. 12). At the construction site, we commonly heard someone exclaim, "No, that box goes below that one" and "No, that box goes under that one because it is bigger." Every step of the process required collaboration, which meant communication, negotiation, and compromise.

> "Every step of the process required collaboration, which meant communication, negotiation, and compromise."

We noticed that children naturally began comparing the size of the boxes, which was a perfect opportunity to discuss measurable attributes of the boxes (e.g. height, width, and even volume). We commonly responded, "I heard you say that box was bigger. What about it is bigger? Is it taller? Or is it wider?" We would then use our hands to demonstrate height and width. "Do you think it could hold more?"

As the buildings grew, the children loved discussing the height of their creations. Comparing and ordering is a natural, critical skill for children (Clements & Sarama, 2007). They frequently speculated about which tower stood tallest. Phrases such as "Ours is twenty-hundred feet tall" were common. When the buildings were completed, each team measured the height of its skyscraper, one partner holding the tape measure at the bottom of the structure while the other read and recorded numbers at the top. Partners reversed positions, remeasured, and compared their results for accuracy. Because many skyscrapers were similar in height, the children could not visually ascertain which was the tallest or shortest. To help them compare, they represented the height of their structures using Unifix® cubes, each cube representing one inch of tower height. The children laid the Unifix towers side by side on the floor for comparison. This allowed them to visually compare the height or length of the towers. After that, the exact numbers seemed less important, and

conversations included such phrases as "Ours is shortest" and "Ours is shorter than Charlie's and Henry's, but taller than Nell's and Olivia's."
Children need a variety of experiences comparing the lengths of objects. They also need experiences that allow them to connect numbers to the lengths of objects. They need opportunities to compare the results of measuring the same object with manipulatives and rulers and to use the manipulative length units to help support the connections between number and length (Clements & Sarama, 2009).

> "Children need a variety of experiences comparing the lengths of objects and connecting numbers to the lengths of objects."

After the skyscrapers were reconstructed on site, the teams added doors and windows. We then asked each group to estimate the number of boxes, windows, and doors before counting them. Estimating was new for the class, so we asked, "About how many boxes (or windows or doors) do you think you used?" We talked about how an estimate is "a best guess." We also discussed whether their guesses were *reasonable* and what that means. We asked them to "try to predict just by looking at the skyscrapers which skyscraper you think used the most boxes. Which one do you think has the fewest boxes?" We documented their predictions and gave each team the task of figuring out and recording how many boxes, doors, and windows the team had used. This gave students the opportunity to count and record with tally marks or numerals the number of materials they had used. This also gave them the opportunity to compare their estimates and predictions and answer the question, "Which group used the most (and fewest)?" To clearly exhibit comparisons among the numbers of objects each group had used, we created graphs, each group recording its data on the appropriate chart by using stickers of boxes, windows, and doors. We spent one day comparing the number of boxes (and another day comparing windows and a third day comparing doors), each time discussing number questions about the graph, such as these:

- "Which groups used more than eight boxes?"
- "Which groups used fewer than three doors?"
- "How many fewer doors did Beckett and Ellie use than Sam and Jack?"
- "Which groups used the same number of windows?"

Discussing comparisons such as these add considerably to children's understanding of number (Van de Walle et al., 2014).

As the children finished their skyscrapers, they worked individually on a smaller-scale city of skyscrapers. Designing these required the use of fine-motor skills, other geometric shapes (e.g., cylinders), spatial visualization, and problem solving. Constructing these allowed the children to create their own three-dimensional structures and solve problems, such as deciding which materials fit in the space provided and how shapes fit together. For these structures, we asked each child to "tell us about your skyscraper. We are going to draw a picture of it without looking, and your job is to tell us what it looks like." Although difficult for some, this activity required each child to communicate about the structure he or she had created and practice using new vocabulary, such as *cylinder*. Students liked this activity so much that they asked us to do it with other structures they created.

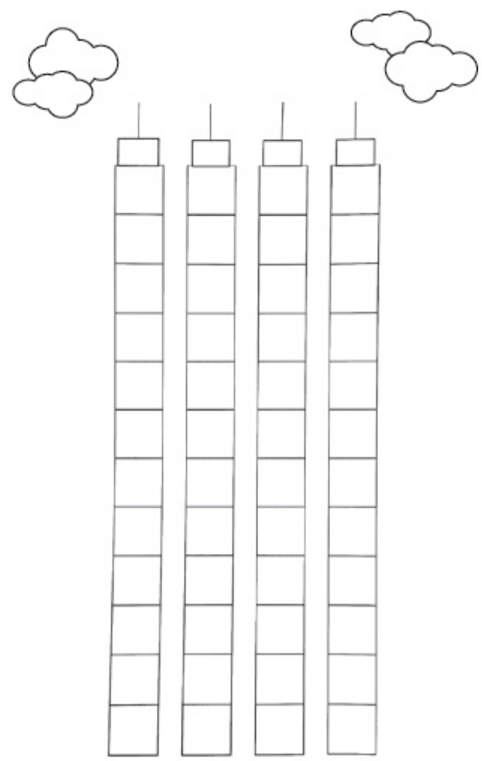
Choice-time activities

The Skyscraper project offers numerous valuable informal and formal opportunities for children to build and create with two- and three-dimensional shapes, using a wide variety of materials. For example, building skyscrapers with Duplos became a favorite choice during free play. More unstructured building offered additional opportunities where children naturally compared the height of their skyscrapers to their height and the height of other people. One child made these direct comparisons with his height, another child's height, and his skyscraper: "It is taller than Eloise, but not as tall as me." He then added, "But, we are still building." After a few minutes, he asked, "Now, is it taller than me?" We took his picture standing next to the skyscraper and showed him the photograph, and he proudly answered that question for himself. He then stated, "But, you are still taller," and kept building. The authors of the Common Core State Standards (CCSSI, 2010) note that making direct comparisons between objects, with a measurable attribute in common, is a foundational kindergarten measurement standard.

The children's interest in building tall structures in the block area continued throughout the investigation. Using large wooden blocks and painter's tape to represent doors and windows, they experimented with balance, shape, and spatial awareness. Building led to more collaboration as taller children helped shorter ones add height to skyscrapers. Teamwork persisted as we challenged them to count windows and doors, giving more-experienced mathematicians an opportunity to help their less-experienced classmates count to higher numbers.

FIGURE 1

To play the Fill the Skyscraper game, students rolled a die and counted the exact number of squares.



In addition to the building activities during choice time, we introduced the Fill the Skyscrapers game (see Figure 1), which we adapted from Van de Walle and his colleagues' (2014) Fill the Towers game. We created game boards with four "skyscrapers." Each skyscraper was a column of twelve one-inch squares. To play the game, children took turns rolling a die and placing the corresponding number of counters on one of the towers. The object was to fill all the skyscrapers with counters. We later introduced the rule that the towers had to be filled exactly, so a roll of a four could not be used to fill three empty spaces. This game created opportunities to assess the children's ability to count and their understanding of number. We often asked them, "How many counters are in that skyscraper?" or "How many more do you need to get to the top?"

Children's literature

In addition to constructing, we read numerous books about construction. *Tonka®: Building Skyscrapers* (Korman,

1999) introduced us to pyramids and taught us why most beams are made of interlocking triangles. We then searched for triangles in all of our construction books. *Look at That Building: A First Book of Structures* (Ritchie, 2011, p. 20) confirmed that "a triangle is the strongest shape," giving us even greater appreciation for the shape. It also showed us how to construct two- and three-dimensional shapes using craft sticks (or straws) and marshmallows. Building with these materials was another favorite choice-time activity and exemplifies how children should "model shapes in the world by building shapes from components (e.g., sticks and clay balls)" (CCSSI, 2010, p. 12). We counted up and down and found examples of triangles, squares, other rectangles, circles, semicircles, and cylinders in books such as *One Big Building: A Counting Book about Construction* (Dahl, 2004), *Construction Countdown* (Olson, 2004), *and When I Build with Blocks* (Alling 2012). The photographs of real skyscrapers in *Amazing Buildings* (Haden, 2003) engaged us; we also admired photographs of Egyptian pyramids, a dome constructed entirely of hexagons, and the sphere-like building at Epcot Center in Florida, which comprises more than 11,000 triangles. We made the books available during choice time for the children to explore; and we urged them to record, by drawing, the different shapes they found.

A powerful learning experience

We worked on the Skyscraper project for four months, which was when our preschoolers' interest waned. However, before we left our investigation, students displayed their work in our Skyscraper Museum, which was open to the public, primarily their families, for a special exhibition. Schoolwork is more meaningful when it is created for a real audience, and so presenting products publicly is a cornerstone of project-based learning (Lamar & Mergendoller, 2010). When students present their work to a real audience, they care more about the quality of their work, and the experience is more authentic. Our students made invitations for the exhibit's opening using their best writing skills. We created documentation panels (i.e., presentation boards containing evidence and artifacts of the children's work), with pictures and descriptions that told the story of the children's learning, so parents could see the process, not just the product. To conclude the project, the children proudly and competently guided their families through the museum, displaying and explaining their work.

> "The project proved to be a powerful learning experience because we recognized and responded to the mathematics that emerged in the children's play."

The project proved to be such a powerful learning experience because we recognized and responded to the mathematics that emerged in the children's play and built on and extended their understandings (Wager, 2013). The mathematics was meaningful because it connected to their play, interests, and everyday activity of building. One of our primary roles as teachers is to observe and help children reflect on and extend the mathematics that arises in their everyday activities, conversations, and play (Clements & Sarama, 2004; Parks & Blom, 2013–2014; Wager, 2013). Project-based learning is a powerful approach to support preschoolers' learning. Preschoolers learn through collaboration and by employing critical-thinking skills as they engage in projects (Bell, 2010). Powerful projects encourage students to explore and investigate their interests and the world around them and experience learning in deep, meaningful ways, which "is the jumping off point to developing students' love of learning and nurturing their natural curiosity" (Bell, 2010, p. 43). By observing our preschoolers' interests and the mathematics within their play and by creating a project on the basis of those interests and significant mathematical ideas, we helped our students work together to build a solid foundation for their mathematical thinking.

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Desmos Art[6]

Nat Banting

If you needed to choose a single tool to add to your math classroom, it would be Desmos—or at least, it should be. From its humble beginnings as an online graphing calculator, it has quickly outpaced all competitors in functionality, usability, and flexibility. It includes a laundry list of connective and intuitive classroom activities, as well as capabilities for flexible control of classroom lessons (see learn.desmos.com). Hundreds, if not thousands of educators are producing classroom materials using Desmos, and many of these are available for public use in a repository curated by their expert staff (see teacher.desmos.com). In short, Desmos has far outgrown the label of free graphing calculator.

That being established, Desmos' basic capabilities as a graphing calculator are powerful in their own right. The ability to easily animate functions, quickly convert them into tables of values, and focus in on critical points make it the most user-friendly graphing calculator for students and teachers alike. For these reasons, I decided to build a Pre-calculus 30 term project (for the very first time) around the software.

I had several goals in mind while building the project. First, I wanted students to become more familiar with the capabilities of the graphing features of Desmos. This familiarity paid off several times during the course, as students reported using Desmos to make sense of questions in their assignments, help them visualize functions and their derivatives in Calculus 30, and even assist older siblings taking mathematics at the post-secondary level. Second, the project addressed several larger goals of Pre-calculus 30. In particular, it provided students the space to connect the properties of the various functions (polynomial, exponential, logarithmic, rational, radical, and trigonometric) with their shapes. It required students to make function transformation decisions based on the desired shape they were trying to re-create, and also provided them a context where domain and range of the above functions gained meaning.

> "The project required students to make function transformation decisions and also provided them a context where domain and range of the above functions gained meaning."

The Pre-calculus 30 course in Saskatchewan, and the corresponding textbook, is largely organized into units that introduce the students to a certain type of function, its properties, its transformation in the Cartesian plane, and, ultimately, has students use this information to solve equations involving the function. This gives the course a very "choppy" feel, and obscures the fact that transformations behave identically across all types of functions. This organization does not lend itself well to a longitudinal project, so I re-arranged the course into four units. The first introduced all of the types of functions and their properties. The second introduced the effect of transformations for all types of functions. The third focused on solving their equations, and the fourth picked up the remaining topics of binomial theorem, permutations and combinations, and trigonometric

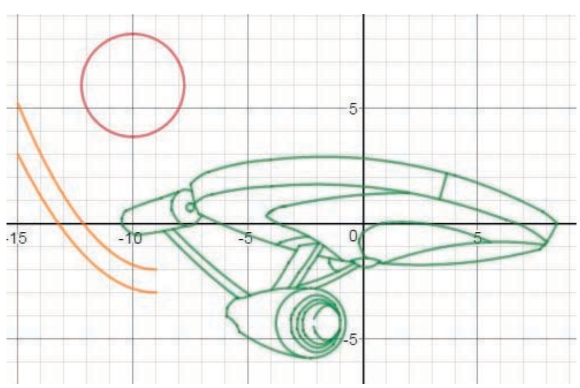
[6] A version of this article was published on January 25, 2016 on Nat's blog at http://musingmathematically.blogspot.ca/2016/01/desmos-art-project.html. Reprinted with permission.

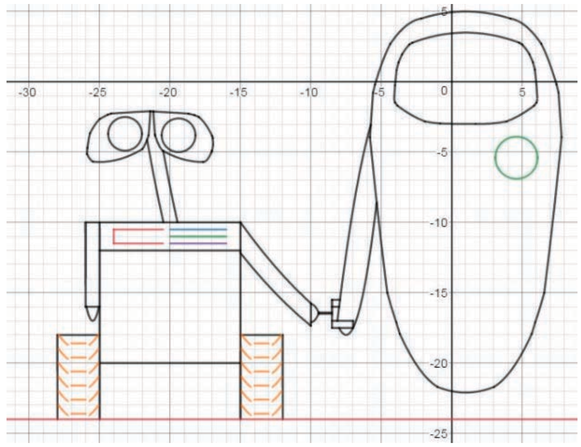
identities. With a significant portion of time now dedicated to functions, their properties, and their transformations, I could use this project to further build a sense of coherence throughout the course outcomes.

I use Desmos regularly in class, so students had a base-level familiarity with the software before beginning the project. I did show them how to restrict domain and range (although most of them stuck exclusively to restricting domain). I introduced the project as we began the unit on function transformations. This gave them about three-and-a-half months to complete it. Throughout the process, I learned several important points with regards to smooth implementation.

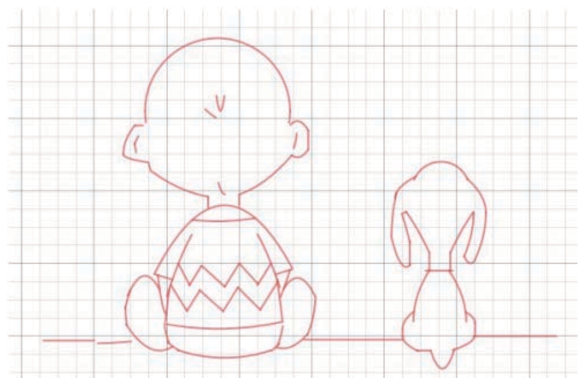
1. It was important that students *copied* a piece of art (this was typically a cartoon of sorts), rather than create an original piece of their own. Requiring that students copy a pre-existing piece meant that they had to think about which functions would accurately model the different portions of the picture they were trying to re-create and how the parameters of the function needed to be transformed to match. If they were to create their own, the model for precision would not exist. In contrast, no lines were arbitrarily chosen in this project.
2. It was important to have the students run their intended re-creation by me prior to starting so that I could mitigate the challenge of the project. Some chose images that were far too elaborate, while others chose images that were far too simple. Some of my colleagues suggested having them draw the shape on paper first, so I included this requirement on the initial project handout. I later decided against this because developing fluidity with the tool was a major goal of the project, and students raised (valid) questions about the purpose of the paper copy.
3. It was important to illustrate how a variety of functions could model the same segment of line, whether that was a curve or line segment. I created an exemplar piece of art that I wanted to re-create, and choose sections to model with functions. Students debated whether certain sections were better modeled as exponentials or polynomials, logarithms or radicals. When I do the project again, I may even have weekly challenges as the students are introduced to a larger variety of functions. For example, I may project a simple image and ask, "What functions would you use to draw this?"

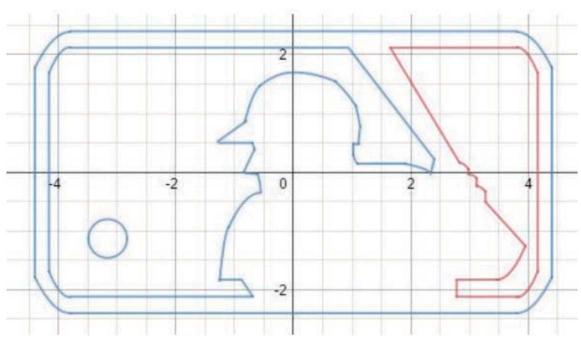
Although I used this project in Pre-calculus 30, it could be adapted for other courses. Students could re-create a piece of art using strictly parabolas (Pre-calculus 20) or line segments (Foundations and Pre-calculus 10). Two examples of parabolic art created by the students of Ontario teacher Mary Bourassa are pictured below (used with permission).

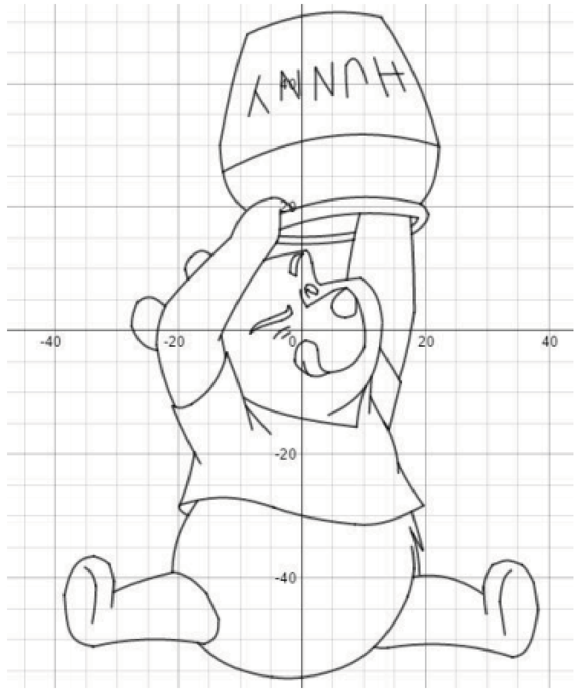




On the whole, the re-creations of this class were fantastically done. I was blown away by the students' precision. The project created an avenue to incorporate Desmos into the entire course, and also provided students with an atypical task as compared to the rest of the course, which was very heavy on symbolic manipulation and computation.

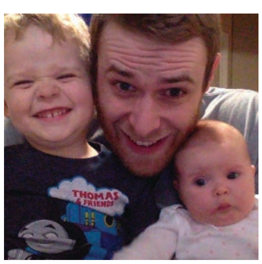






Pre-calculus 30 is an extremely dense course, but I found the Desmos art project to be a way to build familiarity with an amazing classroom tool, to provide a reason to use the concepts of function, transformations, domain, and range with a high degree of precision, as well as to highlight the visual and geometric representations of functions that can be so easily overlooked in a course largely focused on algebraic manipulation.

For more details about the project, including handouts, a tutorial sheet, and more examples of student work, head to Nat Banting's blog, Musing Mathematically:
musingmathematically.blogspot.ca/2016/01/desmos-art-project.html



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Matching Tests: Proof[7]

Jehu Peters

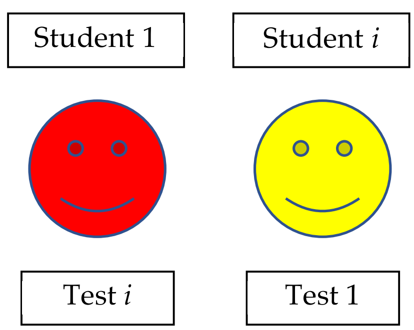
In my last article (*The Variable*, Volume 2, Issue 3), we counted how many ways there are to hand students back their tests so that no student had his or her own test. In this article, I present a proof of the fact that the probability that no student receives his or her own test is always approximately 37%.

It turns out that this type of matching problem has a special name: a derangement. For a group of n students, we denote the derangement as $D(n)$, where $D(n)$ is the number of arrangements of a set of n objects such that no object appears in its original position. From my previous article, we know that $D(3) = 2$ and $D(4) = 9$ (we will use these later). However, to determine this quantity for larger values of n, we need a formula.

Suppose that there are n students. How many ways are there of handing back a test to Student 1? We certainly can't give him his own test. However, we could give him any of the other $n - 1$ tests. Let's say that we give him Test i.

Now, it would be natural to consider how many ways there are of handing back a test to Student 2, Student 3, and so on. However, this becomes very cumbersome and will not easily lead us to the elegant formula that we are seeking.

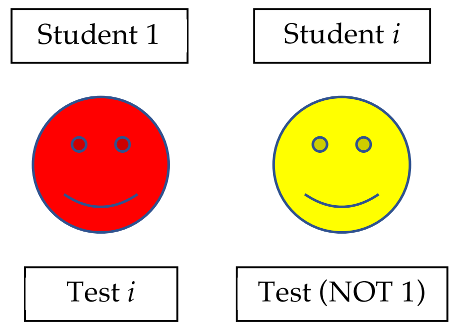
Instead, given that Student 1 has Test i, we will consider how many ways there are of handing out the next test to Student i (the student whose test we just handed to Student 1). We have only two options. In the first case, Student i gets Student 1's test and in the second case, he gets any test other than Student 1's. Here is a picture of the first case:



Now Student 1 and Student i have tests to mark and we have $n - 2$ students who still need tests. However, assigning them tests is analogous to our original problem, and we can denote this as $D(n - 2)$. We will use this result to create our formula shortly.

In the second case, suppose Student i is not permitted to mark Student 1's test. Thus, in this case, Student i could receive any of the other class tests, except Test 1. Therefore, he has 1 forbidden test.

[7] A prior version of this article was published on October 5, 2015 on Jehu's blog at https://themathbehindthemagic.wordpress.com/2015/10/12/matching-tests-proof/. Reprinted with permission.



We then realize that all of the other students also have 1, and only 1, forbidden test (namely, their own). Again, this is analogous to our original problem with $n - 1$ students still needing a test to mark (since we have not given Student i a test yet). We can denote this as $D(n - 1)$.

In summary, to hand out n tests, we have $(n - 1)$ ways for the first test, and 2 cases for the second test (case 1 = $D(n - 2)$, case 2 = $D(n - 1)$). If we multiply these, we get the total number of ways we can hand out the tests. That is:

$$D(n) = (n-1)\big(D(n-2) + D(n-1)\big)$$

We can use this formula to calculate derangements as high as we like, provided we calculate them one after another. For example:

$$D(5) = 4(D(3) + D(4)) = 4(2 + 9) = 44$$
$$D(6) = 5(D(4) + D(5)) = 5(9 + 44) = 265$$

These are exactly the numerators we found when computing the fractions earlier in our problem!

Now, suppose we have 30 students in our class. The formula says that $D(30) = 29(D(28) + D(29))$. However, we do not know $D(28)$ or $D(29)$. This is the fundamental limitation of a recursive formula: you can not calculate the value of a large term without first calculating all of the values leading up to it.

To determine what would happen in an arbitrarily large class, we need a closed form formula for derangements. To create this closed form formula, we first observe the following pattern:

n	$D(n)$	$D(n) - nD(n-1)$
4	$D(4) = 9$	$9 - 4*2 = 1$
5	$D(5) = 44$	$44 - 5*9 = -1$
6	$D(6) = 265$	$265 - 6*44 = 1$
7	$D(7) = 1854$	$1854 - 7*265 = -1$

We can prove the above pattern continues using induction. The base case has been dealt with above. Our inductive hypothesis is:

$$D(n) - nD(n-1) = (-1)^n$$

Consider $n+1$:

$$\begin{aligned} D(n+1) - (n+1)D(n) &= n\big(D(n) + D(n-1)\big) - (n+1)D(n) \\ &= nD(n) + nD(n-1) - nD(n) - D(n) \\ &= nD(n-1) - D(n) \\ &= -\big(D(n) - nD(n-1)\big) \\ &= -(-1)^n \\ &= (-1)^{n+1} \end{aligned}$$

So by the principal of mathematical induction, the result is true for all positive integers. Using this information, we have a new recursive formula for D(n):

$$D(n) = nD(n-1) + (-1)^n$$

This formula suggests that each derangement is calculated like a factorial, but with a +1 or -1 every term. After some trial and error, we find the following formula:

$$D(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

Checking $n = 3$, as $n = 1$ and 2 are trivial, we can see that the base case will hold:

$$\begin{aligned} D(3) &= 3!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) \\ &= 6\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) \\ &= 2 \end{aligned}$$

To prove that the formula is true for all positive integers n, we use induction. Suppose the formula is true for $D(n)$. Consider $D(n+1)$:

$$\begin{aligned} D(n+1) &= (n+1)D(n) + (-1)^{n+1} \\ &= (n+1)n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + (-1)^{n+1} \\ &= (n+1)! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + (-1)^{n+1} \\ &= (n+1)! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + \frac{(-1)^{n+1}(n+1)!}{(n+1)!} \\ &= (n+1)! \sum_{k=0}^{n+1} \frac{(-1)^k}{k!} \end{aligned}$$

Thus, the formula is true for any n. This formula is closed, but complicated. The last step is to relate the derangements to our original problem and to perform an approximation. Since we were looking for a probability, we were really interested in $\frac{D(n)}{n!}$ (the number of possible ways to hand out the tests so no student has their own test, divided by the total number of ways of handing out the tests). Using the Taylor series approximation for e^x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

we get the following:

$$\frac{D(n)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx e^{-1}$$

Finally, we have our answer! Since the number of students determines how many terms of the sequence for e we use, the approximation is valid for even small classes of only 10 or 15 students. Therefore, the probability of handing out tests such that no student receives his or her own test is approximately $e^{-1} \approx 37\%$.

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Reflections *is a monthly column for teachers, by teachers on topics of interest to mathematics educators: reflections on classroom experiences, professional development opportunities, resource reviews, and more.*

An Open Conversation About Curriculum

Sharon Harvey

"When am I going to use this in the real world?"

I don't know if it's because I teach math, or if all teachers are asked this question, but it's the time of year during which it comes up a lot. And this year, for the first time, I engaged in a conversation with the students instead of issuing a witty remark such as, "Well *you* might not, but the smart kids will," which usually garners plenty of chuckles and allows us to move on, forgetting the question that was asked in the first place.

But students *should* question. Why are they learning these particular mathematical concepts, and why right now, and when are they going to be useful? We are trying each day to help students become critical thinkers and problem solvers, but too often shut them down when they inquire about the purpose of a lesson.

So here's what I, and the students, learned from the conversation initiated by this question:

- They didn't know what a curriculum is
- They had no idea that teachers do not just teach from a text book
- They had no idea that teachers have to plan lessons every day
- They didn't know that sometimes, we are as frustrated as they are about what they need to learn.

So we opened a curriculum document. I projected the Aims & Goals of our Saskatchewan Mathematics Curriculum (e.g., Ministry of Education, 2010).

In our curriculum, there are four overarching goals for the teaching and learning of mathematics: Logical Thinking, Number Sense, Spatial Sense, and Mathematics as a

Human Endeavor. I explained that these are what I'm trying to help students achieve, and the material is a vehicle to get them here. I asked them to try to explain if, and when, they felt they were working towards these goals.

> Logical Thinking: *students should develop and be able to apply mathematical reasoning processes, skills, and strategies to new situations and problems. This goal encompasses processes and strategies that are foundational to understanding mathematics as a discipline.* (Ministry of Education, 2010, p. 8)

Student had very little trouble coming up with examples of when this happens. Math class is full of new situations that require previous knowledge to navigate. And as students move forward in their math education, they begin to see the importance of previous units of studies. Some also identified persistence as something they have developed and strengthened in math class as a result of having to solve new problems.

> Number Sense: *students should develop an understanding of the meaning of, relationships between, properties of, roles of, and representations (including symbolic) of numbers and apply this understanding to new situations and problems.* (Ministry of Education, 2010, p. 9)

Again, this goal had pretty strong relevance to and was easily observable in our day-to-day work in the classroom. How many decimal places do we use? When do numbers have equivalency? At what point is it okay to use 2.83 instead of $\sqrt{8}$? We had just been working on solving quadratics, so the idea of an exact value was pertinent. The more they reflected on their previous mathematical work, the more easily they were able to identify when one set of numbers was more appropriate in a given situation than another, or when they would need to move between sets of numbers.

> Spatial Sense: *students should develop an understanding of 2-D shapes and 3-D objects, and the relationships between geometrical shapes and objects and numbers, and apply this understanding to new situations and problems.* (Ministry of Education, 2010, p. 10)

Earlier in the year, students had worked on a problem (Banting, 2016) that involved doubling the surface area of a house. Students were quick to reference this as an example of when they had to use their knowledge about 2-D shapes and 3-D objects and apply it in a new situation.

> Mathematics as a Human Endeavor: *students should develop an understanding of mathematics as a way of knowing the world that all humans are capable of with respect to their personal experiences and needs.* (Ministry of Education, 2010, p. 10)

The students struggled with the language of this one, so we looked a little closer at the list that suggested what an understanding of mathematics as a human endeavor would result in, which includes:

- recogniz[ing] errors as stepping stones towards further learning in mathematics;
- enjoyment, curiosity, and perseverance when encountering new problems;
- self-confidence related to mathematical insights and abilities.

Students had differing reactions to this goal. They said that my classroom was a safe place to make mistakes in that they felt confident that they wouldn't be ridiculed for making a mistake, but they weren't sure that mistakes were used as a way to further their understanding in math. And, most discouragingly, they did not all feel confident in their mathematical abilities: by the time they had reached my classroom, they felt that they knew exactly the extent of their mathematical abilities—and some of them were under the impression that they'd never be any good at it. Those same students also said they didn't believe that I could change this—but I'm surely going to work on trying!

> "Rather than trying to explain when students might use polynomials in the future (they won't), focus on the skills that learning, and mastering, a new concept helps to develop."

As a result of this conversation, my course outlines have changed for next year. These goals are highlighted as the focus of math class. I hope to use this language often in my classes and remind students that *all* of them are capable of achieving these goals… And that logarithms, even if we hate them, can help to strengthen our persistence!

So, my recommendation to you: Rather than trying to explain when students might use polynomials in the future (they won't), focus on the skills that learning, and mastering, a new concept helps to develop, and how those skills are critical to future success. And lastly, please stop telling students that the world they live in isn't real. So often, students hear that we are "preparing them for the real world," or "when you get to the real world…", insinuating that the work they do every day isn't as valuable or as important as the work they will do in the future, or outside the walls of the school. But school *is* real. The work they do every day *is* real. Their world is very real—and it's the world that you've chosen to make a career in.

References

Banting, N. (2016, February 23). My favourite surface area question [Blog post]. Retrieved from http://musingmathematically.blogspot.ca/2016/02/my-favourite-surface-area-question.html

Ministry of Education. (2010). *Foundations of Mathematics 20*. Regina, SK: Author. Available at https://www.edonline.sk.ca/bbcswebdav/library/curricula/English/Mathematics/Mathematics_Foundations_Of_20_2010.pdf

Sharon Harvey has been a teacher within the Saskatoon Public School Division for eight years. She has taught all secondary levels of mathematics, as well as within the resource program. She strives to create an inclusive and safe environment for her students.

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description. If you know about an event that should be on our list, please contact us at thevariable@smts.ca.

Within Saskatchewan

Workshops

Using Tasks in Middle Years Mathematics
August 16, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Using tasks in a middle years mathematics classroom can provide rich opportunities for differentiated learning and authentic assessment. How do we choose tasks that meet both curricular outcomes and student needs? Tasks allow students to enter mathematics where they are at and extend their learning. In this workshop we will look at a variety of resources for finding good middle years tasks. We will also reflect and discuss what planning and teaching moves can assist in maximizing student learning through mathematics tasks.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/using-tasks-middle-years-mathematics-0

Number Talks and Beyond: Building Communities Through Classroom Conversation
August 17, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Classroom discussion is a powerful tool for supporting student communication, sense making and mathematical understanding. Curating productive math talk communities requires teachers to plan for and recognize opportunities in the live action of teaching. Come experience a variety of classroom numeracy routines including number talks, counting circles, quick images and more. Take math conversations to the next level by strengthening your skills as a facilitator of classroom discourse and student thinking.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/number-talks-and-beyond-building

Conferences

Saskatchewan Understands Math (SUM) Conference

October 23-34, Saskatoon, SK
Presented by the Saskatchewan Mathematics Teachers' Society (SMTS), the Saskatchewan Educational Leadership Unit (SELU), and the Saskatchewan Professional Development Unit (SPDU)

This year, the Saskatchewan Mathematics Teachers' Society, the Saskatchewan Educational Leadership Unit and the Saskatchewan Professional Development Unit are partnering to co-ordinate a province-wide conference to explore and exchange ideas and practices about the teaching and learning of mathematics. The Saskatchewan Understands Math (SUM) conference is for mathematics educators teaching in Grades K-12 and all levels of educational leadership who support curriculum, instruction, number sense, problem-solving, culturally responsive teaching, and technology integration, and will bring together international and local facilitators to work in meaningful ways with participants in a variety of formats. This year, SUM is featuring keynote speakers Steve Leinwand of the American Institutes for Research and Lisa Lunney-Borden of St. Francis Xavier University. See the poster on page 4, and head to our website for more information.

Beyond Saskatchewan

MCATA Fall Conference 2017: A Prime Year for Mathematics
October 20-21, 2017, Enoch, AB
Presented by the Mathematics Council of the Alberta Teachers' Association

Join the Mathematics Council of the Albeta Teachers' Association in celebrating their annual fall conference in Enoch, Alberta. This year's keynote speakers are Michael Pruner, a high school mathematics teacher with a Thinking Classroom in North Vancouver and president of the BC Association of Mathematics Teachers, and Sunil Singh, author of the book, *Pi of Life: The Hidden Happiness of Mathematics* and a self- proclaimed Mathematical Jester who is transforming the way mathematics is revealed and discussed all over North America.

See http://www.mathteachers.ab.ca/information-and-registration.html

NCTM Annual Meeting and Exposition
April 25-28, 2018, Washington, DC
Presented by the National Council of Teachers of Mathematics

Join more than 9,000 of your mathematics education peers at the premier math education event of the year! NCTM's Annual Meeting & Exposition is a great opportunity to expand both your local and national networks and can help you find the information you need to help prepare your pre-K–Grade 12 students for college and career success. Classroom teachers, administrators, math coaches, supervisors, college professors, and preservice teachers can all benefit from the sessions and learning at this event. Improve your knowledge and skills with high-quality professional development and hands on activities; gain insights by connecting and sharing with like-minded educators; collect free activities that will keep students engaged and excited to learn; and learn from industry leaders and test the latest educational resources.

See http://www.nctm.org/Conferences-and-Professional-Development/Annual-Meeting-and-Exposition/

Online Workshops

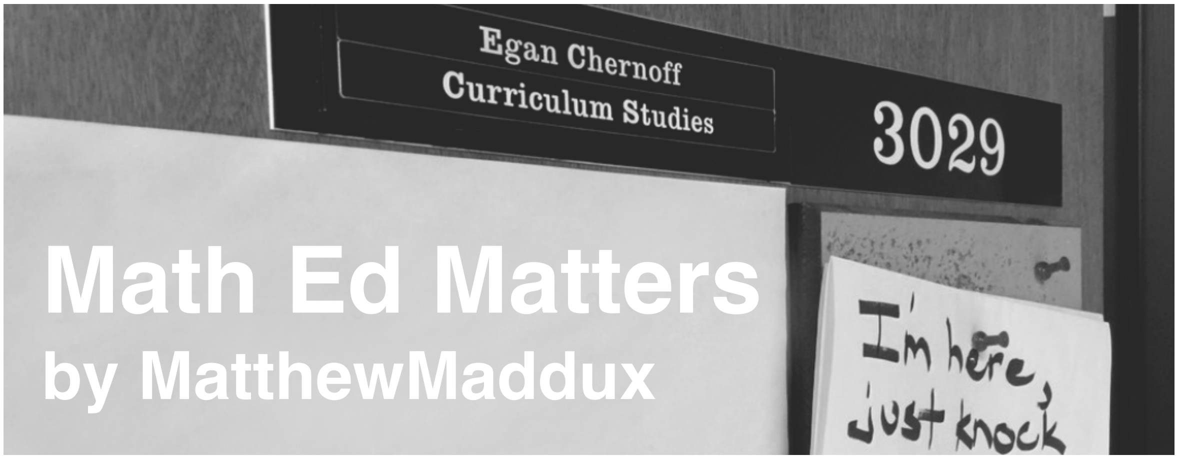
Education Week Math Webinars
Presented by Education Week

Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling and Differentiation.

See http://www.edweek.org/ew/marketplace/webinars/webinars.html

Did you know that the Saskatchewan Mathematics Teachers' Society is a **National Council of Teachers of Mathematics Affiliate**? When registering for an NCTM membership, be sure to support the SMTS by noting your affiliation during registration.

NCTM | AFFILIATE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS



Math Ed Matters by MatthewMaddux *is a bimonthly column telling slightly bent, untold, true stories of mathematics teaching and learning.*

The Lottery is a Tax on…

Egan J Chernoff

The lottery is a tax on the stupid.
Egan plays the lottery.
Therefore, Egan is stupid.

Now that I see it in print, that's rough—even more so, because you could swap *stupid* with *poor* or even *mathematically challenged*! But, as I typed in "the lottery is a tax on…", Google's autocomplete feature gave me three options: poor, stupid, and mathematically challenged. Actually, there was a fourth option of poor *and* stupid.

Full disclosure: As I've alluded to in the above syllogism, I play the lottery. And there you have it, just like that, it would appear that I'm stupid, and poor, and mathematically challenged. Quite the mess I've gotten myself into here. With no immediate plans to turn off Google's autocomplete service on my computers, nor to stop religiously buying $3 Lotto 6/49 and $1 Western 649 tickets (I never play the $1 EXTRA) every single Wednesday and Saturday, my last hope lies in disproving the conclusions of these syllogisms. In other words, it's time to openly test my stupidity.

Good news—maybe I'm not stupid after all. The conclusion of the syllogism that I began this article with is not necessarily valid; as it turns out, I made a sweeping generalization. Consider a different, but similar syllogism:

Birds fly.
The Emu is a bird.
Therefore, Emus fly.

Emus, of course, are flightless birds. They don't fly. Similarly, although *Egan plays the lottery,* maybe *Egan is* not *stupid* after all? To make absolutely sure, let's turn our attention to the well-known aphorism: the lottery is a tax on the stupid/poor/mathematically-challenged.

The Lottery is a Tax on the *Mathematically Challenged*

First, let's address the elephant in the room: You're (probably) not going to win the lottery. You're (probably) not going to win Lotto 6/49. You're (probably) not going to win Western 649. You are definitely (probably) not going to win Lotto Max or Western Max. I am aware of this fact each and every time I play the lottery. How do I know? I've crunched the numbers.

The odds of winning the lottery—I'm talking Jackpot here—for Lotto 6/49 and Western 649 are the same. You have a one in thirteen million, nine hundred eighty-three thousand, eight hundred sixteen (1/13983816) chance of winning the lottery (if you buy just one ticket). For illustrative purposes, let's look at the calculation.

"You're (probably) not going to win Lotto 6/49. You're (probably) not going to win Western 649. You are definitely (probably) not going to win Lotto Max or Western Max. I am aware of this fact every time I play the lottery."

Winning the lottery is simple (if not probable): If the six numbered balls drawn, out of the 49 numbered balls in total, match the six on your ticket, you win. The order in which the balls are drawn does not matter. Say, by way of example, you choose the numbers 4 8 15 16 23 42. Believe it or not, there are 720 (or 6!) different ways that those six numbers can be arranged, but all of those 720 different ways are essentially, according to the rules of Lotto 6/49 and Western 649, the same ticket (i.e., 4 8 15 16 23 42 and 23 16 4 15 42 8 and the other 718 different arrangements are the same numbers, just presented in different order). Keep this number, 720, in mind. Now, there are 49 different numbers that could be drawn first, and once that ball is drawn, it can't be drawn again. So, for the second number that is drawn, there are only 48 left to draw from. Then, there are 47 left for the third ball drawn, and so on. This equates to $49\times48\times47\times46\times45\times44$. "Multiplication?!"

Multiplication of the numbers—as opposed to, say, addition—stems from the Fundamental Counting Principle, which is usually taught in school by way of examples. Say your meal consists of a choice between 3 sandwiches and 2 drinks. Then, there are six possible meals that can be made from the 2 sandwiches and 3 drinks: Sandwich1 with Drink1, Sandwich1 with Drink2, Sandwich1 with Drink3, Sandwich2 with Drink1, Sandwich2 with Drink2, and Sandwich2 with Drink3. (Keep this principle in mind the next time some fast-food company or Dell computers is bragging about how many different choices they have for their customers.) Putting the former and the latter together, there are $(49\times48\times47\times46\times45\times44)/720$ or 13 983 816 possible lottery tickets, with the division by 720 taking into account the fact that a given set of numbers can be arranged in 720 ways. So, if you buy one lottery ticket, you have a 1 in 13 983 816 chance of winning the lottery. Yes, if you buy two tickets instead of one, then you have doubled your chances of winning.

Extending the argument presented for Lotto 6/49 and Western 649 to Lotto Max and Western Max —which also uses 49 possible numbers, but you have to pick seven instead of six numbers—there are $49\times48\times47\times46\times45\times44\times43/5040$, or 85 900 584 possible ticket combinations. Actually, when you buy the ticket you have to play three sets of numbers, which means that your odds of winning the Jackpot are 3 out of 85 900 584 or 1 in 28 633 528. (The numbers associated with winning the lotteries in the United States, e.g., Powerball, Mega Millions, and the like, are even zanier.)

Like I said earlier, you are (probably) not going to win the lottery. As shown, you have a 1/13983816 chance to win Lotto 6/49 and Western 649, and a 1/28633528 chance to win Lotto Max and Western Max. However, as countless math students around the world demonstrate on a daily basis, fluency with calculations or algorithms does not, necessarily, equate with a conceptual understanding of said topic; in this instance, the probability of winning the lottery. Alternatively stated, my ability to determine the odds of winning the lottery does not necessarily mean that I'm not mathematically challenged.

> "As countless math students around the world demonstrate on a daily basis, fluency with calculations or algorithms does not, necessarily, equate with a conceptual understanding of said topic; in this instance, the probability of winning the lottery."

However, at least with respect to the lottery, I can say, with confidence, that I'm not mathematically challenged. I've looked at the numbers from many different angles. I am comfortable with the concepts of dependent and independent events, permutations, combinations, factorials, and other mathematical notions related to the mathematics of the lottery. Digging into the lottery a little deeper, I've played around with the expectation and expected value given various different scenarios involving the size of the jackpot, and have looked at assumptions around how many people buy tickets and potential numbers they might play (e.g., 1 2 3 4 5 6). Beyond calculating the odds of winning the lottery, I have also dug into the research on how we, human beings, have difficulty comprehending very large numbers and very small numbers, such as 1/13983816. Although I'm not sure I should admit it, I've even conducted a though experiment where, given the right conditions, I would buy each and every ticket combination possible, which would guarantee that I won the jackpot. You'll be happy to hear that cooler heads have prevailed, which is a saving-face way of saying that nobody would bankroll my buy-every-ticket solution. To reiterate, with respect to the lottery, I don't think that I'm mathematically challenged.

Yet, there I am, twice a week, buying my lottery tickets. And I'm not alone. Lots of math teachers play the lottery. I even know of math professors who play the lottery. Speaking of university faculty, I regularly—especially at the campus convenience store—observe people that I know, people who are also not mathematically challenged, buying lottery tickets. What gives?! Well, according to Google's auto-search results, another possibility is that we're stupid.

The Lottery is a Tax on the *Stupid*

"Stupid is as stupid does," said Forrest Gump, which, to me, means that you can't judge stupidity based on a person's appearance alone—you have to also take their actions into account. Case in point: I appear, on the surface, to not be stupid. Maybe it's the thick-framed glasses that I had to start wearing at the age of 30, or the plethora of witty math T-shirts that make up a large portion of my wardrobe, or the professorial Birkenstocks. In short, at times, I think I look smart. But, no matter how smart I think I look, I know I'm seen as stupid while standing in line at the gas station with Lotto 6/49 and Western 649 tickets in my hand. You know the look as soon as you see it, another person judging your actions; I get it all the time while buying lottery tickets. But, if they only gave me a chance to explain myself, tell them who I am and

> "No matter how smart I think I look, I know I'm seen as stupid while standing in line at the gas station with Lotto 6/49 and Western 649 tickets in my hand."

what I do, and all the things I know about the lottery, I'd share with them that the lottery only *appears* to be a tax on the stupid.

Let me explain. I contend that the lottery only *appears* to be a tax on the stupid because, as we are finding out through research in various academic fields (e.g., psychology), we need to rethink our notion of stupid. The field of economics, thanks largely to Amos Tversky and Daniel Kahneman, has, for the most part, long abandoned the notion of the rational economic man, and a plethora of research detailing various heuristics and biases has developed into a field known as behavioral economics. As an example, let's say you are playing a very simple lottery where all you have to do is pick one number from, say, 49 numbers. If your number is drawn, then you win the jackpot; if it's not, then you lose. Simple, right? Here's the rub: What if I told you that a particular number has not been drawn for a long time—a really long time? Well, should you decide to start playing that particular number because it's "due," you, my friend, are falling prey to what is known as the gambler's fallacy. In the case of the lottery, and in many other instances (e.g., flipping a fair coin), if a number (or face of a coin) has not come up for a long time, it does not have a higher chance of coming up in the future. The thing is, though, it kind of *feels* like it should be more likely to come up. And you're not alone if you feel this way. A lot of people fall prey to the gambler's fallacy, even after being told about it. This fallacy can account for why, if you're ever in a casino, you'll see some chairs at the slot machines tipped up against the machine, which are meant to indicate that the people who were playing need a quick break but want to keep playing those particular machines when they get back because, in their head, they're due to pay out soon. Don't you dare start playing their machine! I contend that it is sanctimonious to dismiss all these people as stupid; after all, psychological insights into human behavior can explain why all those chairs are tipped up against the slot machines that seem overdue for a payout.

> "Sure, playing the lottery, knowing full well that you (probably) will not *win*, is admittedly a pretty good definition of stupid. But, if there are so many people involved in the act, there must be something more going on…"

Let's get back, now, to rethinking stupid with respect to the lottery. Sure, playing the lottery, knowing full well that you (probably) will not *win* the lottery, is admittedly a pretty good definition of stupid. But, if there are so many people involved in the act, there must be something more going on than "everybody who plays the lottery is stupid." We could, here, use a method of analysis where psychological insights into human behavior are applied in order to explain lottery ticket decision-making. We could also, relatedly, delve into the well-established "Are humans innate statisticians?" discussion. However, instead, I contend that there are even more very non-stupid decisions being employed by people who play the lottery. Let's look at a few.

I have found, among those people that play the lottery, that you are considered more stupid, and more mathematically challenged, if you always play the same numbers instead of choosing different numbers every time (by using the Quick Pick generator, or some other method). I agree, you don't have a greater chance of winning the lottery if you play the same numbers or, for that matter, different numbers. However, there are pros and cons to doing so. On the one hand, playing the same numbers is a curse because, as anyone who plays the same numbers will tell you, the numbers are forever burned into your brain (e.g., 4 8 15 16 23 42). To be clear, knowing your six numbers isn't the burden. The burden is if you ever miss playing a draw! Oh my god—the anxiety is unreal. Why? Well, imagine if, this one time, the one time you missed playing a draw, your numbers—the six that you play over and over, and which are forever burned into your brain—hit. All you would have to do is look in the newspaper or online and your winning ticket, the one you did not buy, would be staring back at you. You would be devastated! On the other hand, playing the same numbers is a blessing. There is no decision for you to make when you buy your ticket, as it's the same numbers every time. Also, checking the draws is super easy, because you know your numbers off by heart; moreover (for all you conspiracy theorists out there), you don't have to rely on technology scanning your winning ticket and falsely telling you it's not a winner. Adding fuel to the fire, if you just play a random draw of numbers, the Quick Pick, and you forgot to buy a ticket, you would have no way of knowing whether the ticket you would have bought would have won the lottery. Moving beyond the numbers you play, it is also a non-stupid decision to play the lottery from a marketing perspective.

As we all start to get a peek behind the curtain of advertising—thanks, for example, to CBC Radio's *Under the Influence*—we find that we are all susceptible, no matter how much we think we are not, to advertising. One day you're just watching TV or listening to the radio or reading a magazine and then, a few weeks later, you open your fridge and there, for some reason, is a wheel of Laughing Cow cheese spread. Maybe you don't even remember watching a Laughing Cow commercial. You surely don't remember watching it and then saying to yourself, "Hmm, I really need to pick up some Laughing Cow cheese spread the next time I'm grocery shopping." Nevertheless, there it is, sitting in the fridge, staring back at you. Given the number of places that you see the lottery advertised as compared to, say, The Laughing Cow, it is no wonder that people play the lottery. You can't watch TV, listen to the radio, open a newspaper, open a magazine, surf the internet, or read a huge billboard while driving in your car without stumbling across multiple advertisements for the lottery. And let's not forget old and new Lotto 6/49 slogans:

- *You don't just buy a ticket. You play it.*
- *Hey, you never know!*
- *Imagine the freedom.*
- *Always be nice to people who play Lotto 6/49.*
- *Welcome to cloud (64)9.*

Given the power of advertising, coupled with the extreme excess and variety of advertisements, playing the lottery, to me—granted, the guy with a wheel of Laughing Cow cheese spread sitting in his fridge—is a non-stupid decision. Something else, something bigger, is going on here; to paraphrase Pickford (the great Matthew McConaughey) in *Dazed and Confused*, "What can I say? It's beyond me."

I would be remiss not to mention one last point demonstrating that playing the lottery is not, necessarily, a tax on the stupid. The point is that, for the most part, people are not buying a Lotto 6/49 ticket to win the lottery. Of course, they *hope* that they do, but even if they don't, I think they are getting a pretty good bang for their buck when they buy a ticket, as part of the non-stupid decision to buy a lottery ticket is the opportunity to imagine what it would be like to win. In short, it's escapism. Now, before you bash this particular version of it, let me remind you that our lives are full of escapism. Lots of people will tell you that they enjoy going to the movies because they like to suspend their reality for a few hours. Taking a vacation? Same idea. In

"For the most part, people are not buying a Lotto 6/49 ticket to win the lottery. Of course, they *hope* that they do, but even if they don't, I think they are getting a pretty good bang for their buck."

fact, you might distract yourself in a number of ways from the daily grind through entertainment, or fantasy, or simply by getting into your head. Why, then, are we so hell-bent on berating somebody who, for a paltry amount of money, is able to conduct a thought experiment about how their life would be different if they were to suddenly come upon a huge sum of money? As I calculated above, the chances of winning the lottery are in 1 in 13 983 816. But, by the way, if you don't buy that ticket, you have a 0% chance of winning. No escapism for you.

> "We live in a world of $5 coffees and (nearly) $10 pints of beer. Is it really so wrong if I spend a few bucks each week so that I can distract myself for a few moments while I'm stuck in traffic? To me, the answer is no."

We live in a world of $5 coffees and (nearly) $10 pints of beer. A world where people are engaging with each other as avatars in the online world for hours and hours on end (e.g., *World of Warcraft*). A world of furries (men and women who wear animal fur costumes in public), bronies (adult male fans of the children's show *My Little Pony*), and innumerable other crazy things that I don't know about. Considering this new world order, is it really so wrong if I spend a few bucks each week so that I can distract myself for a few moments while I'm stuck in traffic by imagining what it would be like to drive a nicer car, if I could just hit the jackpot? To me, the answer is no. After all, I can afford to play the lottery.

The Lottery is a Tax on the *Poor*

The lottery is a regressive tax—that is, the tax is a greater burden on those who earn less money. To investigate this aphorism further, let's now take a look at income tax brackets and income tax rates, which I will refer to as lottery tax brackets and lottery taxation rates. Knowing which lottery tax bracket you fall in, as well as your lottery taxation rate, should give you a better sense of whether or not you should be playing the lottery.

Those of who are you familiar with Canadian lotteries know that it has become more and more expensive to play over the past number of years. While it used to cost just a dollar to play Lotto 6/49, one dollar became two dollars, and then, in September of 2013, two dollars became three dollars. I'll be using this $3 Lotto 6/49 ticket price for my calculations. With 52 weeks and two draws a week, there are, in total, 104 draws and opportunities to play the lottery each year. $3 (per draw) times 104 (draws) comes out to $312 dollars per year. But wait, we're not done investing in the lottery just yet. Instead of adding the $1 EXTRA on the purchase of my Lotto 6/49 ticket, I use it instead towards the purchase of a Western 649 ticket. Playing the lottery, then, costs me $8 every week (for now). Over the period of one year, I can play the lottery for a grand total of $416. Let's keep this number, $416, in mind as we move on, but also recognize that this amount of money means different things to different people.

> "Let's establish how punishing things really are for an idiot, or a family of idiots, who plays the lottery."

Okay, I'll bite: I'll consider buying lottery tickets as a tax—that is, a burden or a punishment. Looking at the lottery from this perspective, let's establish how punishing things really are for an idiot, or a family of idiots, who plays the lottery. Using approximate numbers from a few years ago, the average income for a family of two or more people in Canada is about $80 000, which means they'll pay $12 862 in income taxes in a year, which works out to a rate of 16%. What if, though, this average income family decides to start investing in the lottery? Using the same numbers, the Lotto 6/49 tax rate works out to less than one half of one percent ($312/$80000 = 0.39%). The Western 649 tax rate, for the same family, works out to around one half of one half of one half of one percent ($104/$80000 = 0.13%). Combined together, the lottery tax rate works out, roughly, to one half of one percent ($416/$80000 = 0.52%). As a rate, not too shabby.

Of course, the key here is that the rate is one half of one percent for a family that is making $80 000. Obviously, the rate is different for someone living closer to the low-income cut-off (Canada's way of saying the poverty line), which, for two people, is just shy of $30 000. The lottery tax rate for this couple is 1.4%. In contrast, a couple earning around $125 000 (which puts you into one of the wealthiest brackets in Canada) would be paying a lottery tax rate of one-third of one percent (0.33%). For a couple who just barely made it into the top income tax bracket in Canada, which means that they make around $203 000, the lottery tax rate is even lower, at around two-tenths of a percent (0.2%). A breakdown of income tax rates and lottery tax rates for different income tax brackets in found below in Table 1.

Income Brackets (Descriptor)	**Income Tax**	**Income Tax Rate**	**Lottery Tax**	**Lottery Tax Rate**
$3000 (Poverty line, two people)	*$4500*	15%	*$416*	1.4%
$45916 (Ceiling of lowest income tax bracket cut-off)	*$6887.40*	15%	*$416*	0.91%
$91831 (Ceiling of next highest income tax bracket cut-off)	($6887.40 + $9412.58) *$16299.98*	17.75%	*$416*	0.45%
$142353 (Ceiling of next highest income tax bracket cut-off)	($6887.40 + $9412.58 + $13135.72) *$29435.70*	20.68%	*$416*	0.29%
$202800 (Ceiling of next highest income tax bracket cut-off)	($6887.40 + $9412.58 + $13135.72 + $17538.33) *$46974.03*	23.16%	*$416*	0.21%
$275000 ($72200 into the highest income tax bracket cut-off)	($6887.40 + $9412.58 + $13135.72 + $17538.33 + $23826.00) *$70800.03*	25.75%	*$416*	0.15%

Table 1. Income and lottery tax rates and brackets.

In Canada, income tax brackets are set by the federal government. And, to nobody's surprise, the percentages change over time. As for lottery tax brackets, which I've compared to income tax brackets in the table above, these too change with fluctuations in ticket price. For the moment, though, the lottery tax rate hovers at around less than one and a half percent for most.

Looking at these numbers raises a number of questions. Should a person or family hovering around the poverty line be playing the lottery? Well, the short answer is that it's up to them. Should a person or family making "six figures" be playing the lottery? Again, it's up to them. However, no matter how you slice it, it's true that the more money you make, the easier—in a financial sense—it becomes to play the lottery.

As mentioned at the beginning of this section, the lottery is a regressive tax—that is, the tax is a greater burden on those that make less money. But what if we shift the focus away from the poor? This means that as you move away from the poverty line, playing the lottery becomes less and less of a financial burden. So, the next time you hear someone whining about how much income tax they have to pay every year, remind them of this good news: As you climb your way through the ranks of the income tax brackets and your rates get higher, your lottery tax rates get lower. You should also remind them that paying taxes isn't necessarily a burden.

The Lottery is a *Tax* on…

For many (especially for our neighbours south of the border), the word "tax" has a negative connotation. As such, for many, "tax" is synonymous with "punishment" or "burden." This negative view of taxes is, I contend, the one represented in the aphorism, "Lotteries are a tax on the stupid (or poor or mathematically-challenged)." In other words, if you're stupid (or poor, or mathematically-challenged), which is allegedly at the root of your decision or predilection to play the lottery, then you should be punished—that is, taxed. As Noam Chomsky discusses in his movie *Requiem for the American Dream* (available on Netflix):

> One place you see it strikingly is on April 15. April 15 is kind of a measure—the day you pay your taxes—of how democratic the society is. If a society is really democratic, April 15 should be a day of celebration. It's a day when the population gets together to decide to fund the programs and activities that they have formulated and agreed upon. What could be better than that? You should celebrate it.
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> It's not the way it is in the United States. It's a day of mourning. It's a day in which some alien power that has nothing to do with you is coming down to steal your hard earned money—and you do everything you can to keep them from doing it. That's a measure of the extent to which, at least in popular consciousness, democracy is actually functioning. Not a very attractive picture.

Maybe it's because I'm Canadian, maybe it's due to the conversations in the household I grew up in, or maybe it's something else; nevertheless, I do not have an issue with paying taxes. And I pay all kinds of taxes. For example, there's income tax and sales tax; add to that my contributions to the Canada Pension Plan, Employment Insurance, and Health Care (which, for me, are central to who we are as Canadians). And we're not done yet. As I am fortunate enough to own a home, I also pay property tax, which is considered to be a wealth tax. The category of wealth taxes also includes gift taxes, estate taxes, and profit taxes. Oh yeah—there's also international taxation, that is, taxes based on any income you might be earning in the global marketplace. Don't forget the taxes associated with your vehicle, as well as the fuel you put into it to make it

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run. In line with my previous column (which focused on my visits to the liquor store), I should also mention the excise or "sin" taxes on alcohol, cigarettes, and, in the very near future, marijuana. No matter how you feel about taxes, buying a lottery ticket is just a drop in the bucket at this point.

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And what of the bucket? The more income you make, the more you pay of each of the different types of taxes listed above. For example, if you make more money, you might own a bigger or better home, which means that you will pay more in terms or property taxes. Don't forget to add wealth taxes (i.e., gift, estate and profit taxes) and international taxes to the mix. Making more money means that you might be in a privileged position to make some investments. Should you make some money selling a rental property or some investments which were listed on the NASDAQ or TSX, expect to pay taxes. Continuing with this line of reasoning, if you make more money, then you might drive a more expensive car and (for reasons unbeknownst to me) fill it with premium gasoline. Maybe you make enough money so that instead of buying the equivalent of four bottles of wine in a reasonably priced box, you buy four individual bottles, which is going to cost you more in terms of the sin tax. As for all the purchases you make with your hard-earned money, yup, every time you make a purchase, you are paying the sales tax. No matter how you slice it, the more money you make and, relatedly, the more money you spend (should you spend it), the more you pay in taxes. According to various sources, Canadians paid approximately 42% of their income in taxes in 2016! Considering this, adding the measly 1.4% or lower tax of buying a lottery ticket every week so that you can daydream about how much better your already great life could be may seem, dare I say, sensible.

The Lottery is a *"Tax"* on the *"Stupid/Poor/Mathematically-Challenged"*
Let me reiterate: You're (probably) not going to win the lottery. But, I really think it's time that we came up with a different aphorism. After all, having looked at the idea that the lottery is a tax on the stupid/poor/mathematically challenged from a number of different angles, to me, the aphorism doesn't hold water. Many numerate people are playing the lottery. It's not (necessarily) because they're stupid. Who can blame them if they're just looking for a quick respite from the daily grind? With a different view of taxes, and given the amount Canadians are already paying, adding a few bucks a week for a lottery ticket is not going to make or break most people financially. If you happen to make a lot of money, indulge in the fruits of your labour! That is, enjoy some regressive taxation by playing the lottery (and, by the way, if you did win, you would not pay taxes on your winnings!). You should also keep in mind that here in Saskatchewan, part of the money spent goes into the Saskatchewan Lotteries Trust Fund, which is the main fundraiser for more than 12,000 sport, culture and recreation groups in the province. Alternatively, the money could be going to a huge, "soulless," multi-national corporation—which brings me to my proposal for a new, but related new aphorism.

> "Having looked at the idea that the lottery is a tax on the stupid / poor / mathematically challenged from a number of different angles, to me, it doesn't hold water."

While my financial status has changed over the years, there have been two constants. First, my religious purchasing of lottery tickets. Second, I have never paid a cent in credit card debt. Believe me: poor or not-poor, buying lottery tickets is much, much easier than paying off credit card debt in full. To me, paying 20% interest on purchases and nearly 25% interest on cash advances is insane. Period. Consider making the minimum payments on a $1000 purchase on your credit card with, say, a rate of 20% with minimum payment options of 2% or $10. Yup, that's right, it would take you 317 months to pay off your debt, and your purchase would accumulate to around $3100 in interest charges payable to MasterCard, VISA, Discover, American Express, and the like. And you say that lotteries are a tax on the stupid/poor/mathematically-challenged? Give me a break. I suggest, as a potential new aphorism, minimum credit card payments are a tax on the stupid/poor/mathematically-challenged. Futher, predatory lending is a tax on the stupid/poor/mathematically-challenged.

Maybe it's time to focus more on teaching financial education in the mathematics classroom. In the interim, let me buy my lottery tickets in peace. After all, isn't the lottery just a tax on the willing?

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