The Variable

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Saskatchewan Mathematics Teachers' Society

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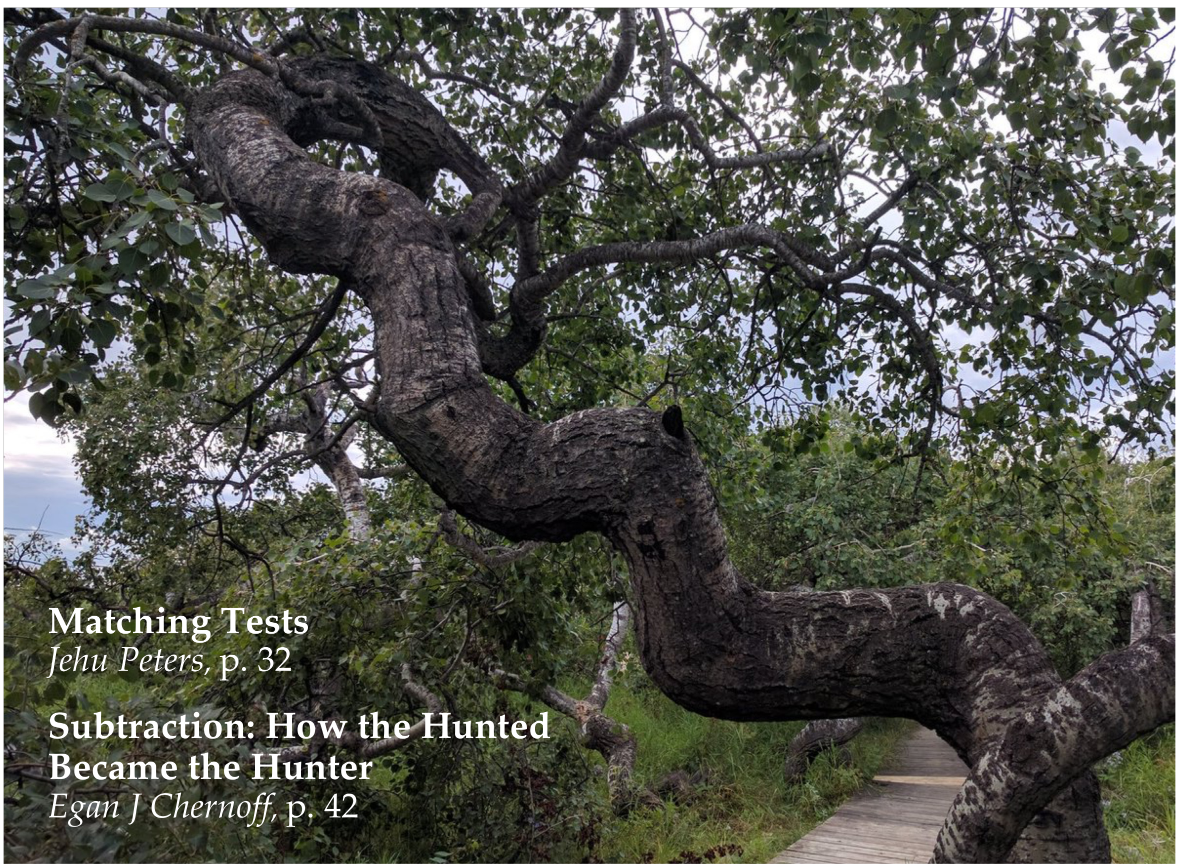

Saskatchewan Mathematics Teachers' Society

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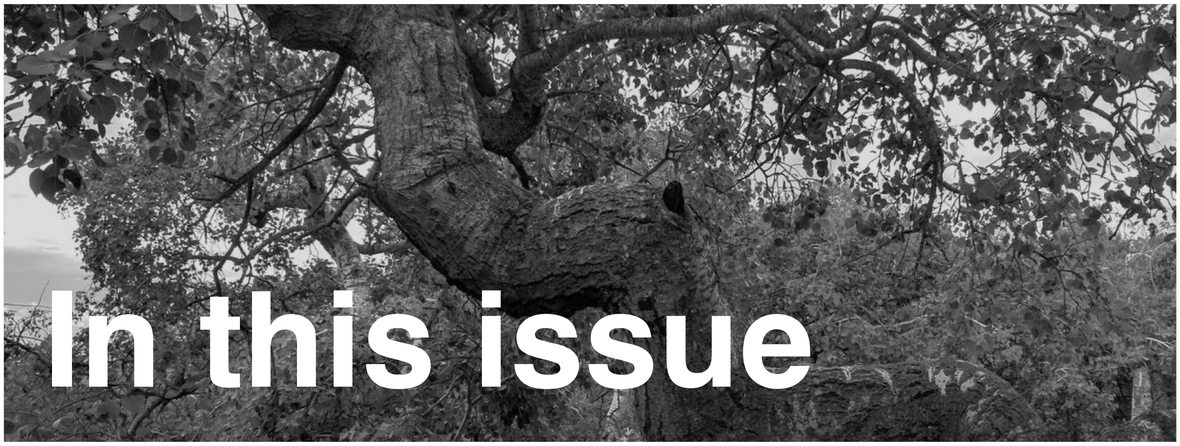
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS



In this issue

Cover Image

This month's cover photo was submitted to the 2016 Saskatchewan Math Photo Challenge by Sarah Thompson. The photo was taken at what is known as the "Crooked Bush" located northwest of Hafford. Every aspen tree in the small grove is twisted in mysterious forms.

SMTS Executive (2016-2018)

Notice to Readers

The views expressed in *The Variable* are those of the author(s), and not necessarily those of the Saskatchewan Mathematics Teachers' Society. In the publishing of any works herein, the editorial board has sought confirmation from all authors that personal quotes and examples have been included with the permission of those to whom they are attributable.

Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants, teacher educators, researchers, and students of all ages, although we encourage Saskatchewan teachers of mathematics as our main contributors. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Articles should be limited to 3000 words or less; authors should also include a photo and a short biographical statement of 75 words or less. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Call for Contributions

Did you just teach a great lesson? Or maybe it didn't go as planned, but you learned something new about the complexities of teaching and learning mathematics. Maybe you just read a book or attended a workshop that gave you great ideas for presenting a topic your students have always found difficult, or that changed your perspective about some aspect of teaching. **Why not share your ideas with other teachers in the province—and beyond?**

The Variable is looking for contributions from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants and coordinators, teacher educators, and researchers. Consider sharing a favorite lesson, a reflection, an essay, a book review, or any other work of interest to mathematics teachers in Saskatchewan. If accepted for publication, your piece will be shared with a wide audience of mathematics educators in Saskatchewan and beyond.

We are also looking for student contributions in the form of artwork, stories, poems, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom and their school, and for teachers to recognize their students' efforts during their journey of learning mathematics.

All work is published under a Creative Commons license. If you are interested in contributing or have any questions, please contact us at thevariable@smts.ca.

We look forward to hearing from you!

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Professional Growth Partnerships
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Leading Together
The SMTS, SELU and SPDU presents ...
#SUM2017
Who: K-12 teachers, coaches, consultants, coordinators, superintendents and directors
Where: TCU Place, Saskatoon
When: October 23-24, 2017
Cost: $315 (early registration) | $375 (regular)
Keynote Presenters
Lisa Lunney Borden, St. Francis Xavier University
Steve Leinwand, American Institutes for Research
Featured Presenters
Kristin Gray, Cape Henlopen School District, Illustrative Mathematics
Glenn Waddell, The University of Reno, Desmos Fellow
Egan Chernoff, University of Saskatchewan
Gale Russell, University of Regina
NCTM
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Saskatchewen Understands Mathematics Conference 2017

Keynotes

Steve Leinwand

Practical Suggestions for Building a Powerful and Professional 2017-2018 To-do List

This fast-paced and example-laden pep-talk will discuss and model a set of instructional shifts that NCTM's Mathematical Teaching Practices and the quest for more effective instruction require us to consider in order to enhance our teaching and our students' learning.

Featured Session 1: Proving the Leadership Necessary for Making Mathematics Work for All Students (Part 1)
We know that effective programs of K-12 mathematics require informed and effective leadership. This part one of a two-part series of workshops will focus on specific understandings that every mathematics leader needs to have about effective mathematics programs, with a focus on high quality instruction, to be in a position to advocate for and support such programs.

Featured Session 2: Proving the Leadership Necessary for Making Mathematics Work for All Students (Part 2*)
We know that effective programs of K-12 mathematics require informed and effective leadership. This part two of a two-part series of workshops will focus on specific strategies and initiatives that every mathematics leader needs to establish, nurture and monitor to ensure that the effective mathematics programs discussed during part one are available to all students in every school. We'll take particular look at a range of collaborative structures that reduce professional isolation and support professional growth.

*attendance at Part 1 is not required to participate in Part 2

Lisa Lunney Borden

The Role of Mathematics Education in Reconciliation

The 2015 TRC final report that includes calls to action in response to the horrors of residential schools for Aboriginal Canadians that are focused on establishing a renewed relationships between non-Aboriginal and Aboriginal Canadians to "restore what must be restored, repair what must be repaired, and return what must be returned" (2015, p. 6). The TRC names the education system as having an essential role in repairing the damages caused by residential schools. Lisa will reflect on her 22-year career as a teacher and researcher working in Indigenous communities, primarily Mi'kmaw communities, to explore the role of mathematics education in reconciliation. She will share stories of hope and healing that have emerged through the Show Me Your Math program, inquiry projects, outreach programs, and teacher professional learning that give insights into how mathematics can aid in reconciliation.

Featured Session 1: Our Ways of Knowing: Teaching Math with Verbs and Space
Lisa will share a model for considering ways in which Indigenous languages, community values, ways of knowing, and cultural connections can impact mathematics learning for Indigenous learners. Participants will go more deeply into the pedagogical implications of this model that are linked to the ways of knowing that emerge from an understanding of the structure of Indigenous languages. We will engage in tasks that highlight the value of verbifying and spatializing mathematics teaching and learning. Examples will be drawn from Kindergarten to Grade 12 to highlight how these approaches span all levels.

Featured Session 2: My Elders were Mathematicians Too: The Value of Culturally-based Inquiry
Lisa will share the story of Show Me Your Math, a program that invites Indigenous students in Atlantic Canada to explore the mathematics that in inherent in community ways of knowing, being, and doing. She will share the history of this program, how it has changed over time to focus more on inquiry, and how it might be developed in other regions. We will explore examples of projects that have been completed, examine the benefits of these projects and discuss how such projects help to restore, reclaim, and return community knowledge that has been eroded by colonialism.

For more information on keynote and featured speakers, please visit:
http://smts.ca/sum-conference/sum-keynote-presenters

Message from the President

Happy first birthday to *The Variable*! It's hard to believe that it has only been a year. This "little" *nouveau format* journal already feels so established, so much so that it's hard to remember life before its creation. And yet, I remember that at the meeting when it was first imagined, a certain executive member doubted that we'd be able to sustain it. While 2017 did prompt a reduction in issues, that member was clearly crazy to doubt the commitment of our amazing editor and vice-president—Ilona—and her equally amazing team. I'm so privileged to get to travel the province for my work, and have heard great feedback in all corners of the province from math educators who read and look forward to every edition. I knew the team really struck that magical balance between practical, theoretical, and playful when a very respected literacy consultant I know remarked, "I love that thing! I share it with everyone I know!" When the literacy folks are reading and sharing your math journal, you know you're doing more than alright. So please join me in thanking Ilona, our new co-editor Nat, and the rest of the *Variable* team for all their hard work in putting this periodical together. I know it enriches the teaching of mathematics (and literacy!) educators within and far beyond the borders of Saskatchewan.

May also brings us to our annual strategic planning. While 2017 has marked itself as a very difficult year to be an educator in Saskatchewan, I am reminded daily of the amazing work and dedication of the teachers in this province. The SMTS executive is going into our planning cycle with you, and all dedicated, hardworking, (exhausted) teachers in mind. How might we make 2017-18 even just the tiniest bit better for you? How can we best be a voice for mathematics educators in this province? How can we provide you with a diversity of options to support your teaching, and how do we make sure these options meet your needs? How do we support you in bringing joy, play, and learning into your math class? We, the SMTS, are committed to being a voice and a support for teachers during these trying times, and perhaps even more so because of them.

Of course, we hope that you will able to join us in the fall to celebrate and learn together at the Saskatchewan Understands Math (SUM) Conference 2017. But if not, fear not—you will continue to hear from us in *The Variable*, via our website, on Twitter, and wherever else we can talk about the teaching and learning of mathematics in this province. If you'd like to join us, you're in luck: it's an election year. Consider joining our executive! Or a committee! Maybe you'd like to relieve me from writing these President's Messages? (Bueller?) In whatever capacity you'd like to join in the conversation, we'd love to have you. As always, it's together that we do our best work.

We look forward to hearing from you. Until then, stay mathy, friends, and remember that your work is deeply valued and appreciated.

Michelle Naidu

Problems to Ponder

Welcome to the May/June edition of Problems to Ponder! This collection of problems has been curated by Michael Pruner, president of the British Columbia Association of Mathematics Teachers (BCAMT). The tasks are released on a weekly basis through the BCAMT listserv, and are also shared via Twitter (@BCAMT) and on the BCAMT website.

Have an interesting solution? Send it to thevariable@smts.ca for publication in a future issue of The Variable.

British Columbia Association of Mathematics Teachers

I am calling these problems 'competency tasks' because they seem to fit quite nicely with the curricular competencies in the British Columbia revised curriculum. They are non-content based, so that all students should be able to get started and investigate by drawing pictures, making guesses, or asking questions. When possible, extensions will be provided so that you can keep your students in flow during the activity. Although they may not fit under a specific topic for your course, the richness of the mathematics comes out when students explain their thinking or show creativity in their solution strategies.

I think it would be fun and more valuable for everyone if we shared our experiences with the tasks. Take pictures of students' work and share how the tasks worked with your class through the BCAMT listserv so that others may learn from your experiences.

I hope you and your class have fun with these tasks.

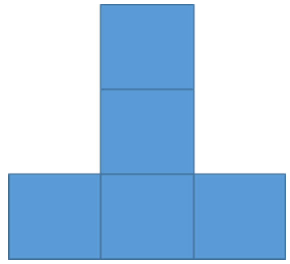
Michael Pruner

Primary Tasks (Kindergarten-Intermediate)

What is the Shape?[1]

A shape is made with linking cubes. When you look at it from one side, it looks like this:

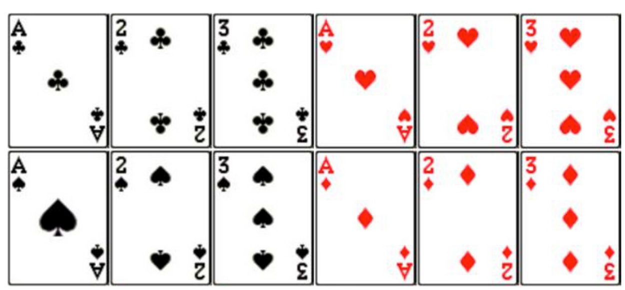
[1] Small, M. (2012). *Good questions: Great ways to differentiate mathematics* (2nd ed.). New York, NY: Teachers College Press.



What might the structure look like?

Five Cubes[2]
Using exactly 5 interlocking cubes, make as many shapes as you can so that all five cubes are touching the table. How many *different* shapes can you make?

Making Ten
Each group of students has a set of playing cards from ace to three (aces have a value of 1). Use these 12 cards to make the numbers from 1-10, using any operations you like.



Intermediate and Secondary Tasks (Intermediate-Grade 12)

Sharing Bacon[3]
You are a chef at a summer camp and you are frying 30 identical strips of bacon for this morning's breakfast. A counselor comes in to inform you that there are only 18 campers

[2] Spring 2011 problem set. (2011). *Vector, 52*(1), 47-49.

[3] Mason, J., Burton, L, & Stacey, K. (1985). *Thinking mathematically*. Essex, England: Prentice Hall.

coming in for breakfast and they all love bacon. What is the minimum number of cuts necessary? What is the minimum number of pieces?

Extensions: How do you know it is the minimum? What about sharing amongst 17 campers? 16 campers? n campers?

Magic Squares [4]

A magic square is a square grid with n rows and n columns, filled with distinct numbers from 1 to n^2, such that the sum of the numbers in each row, column, and both long diagonals is the same.

1. Can you come up with a 2×2 magic square?
2. What about a 3×3 magic square?
3. What value does each row, column, and long diagonal need to sum to in a $n \times n$ magic square?

Extensions: Investigate magic rectangles and magic triangles.

Box of Marbles [5]

In a box, you have 13 white marbles and 15 black marbles. You also have 28 black marbles outside of the box.

Remove two marbles, randomly, from the box. If they are of different colours, put the white one back in the box. If they are of the same colour, take them out and put a black marble back in the box. Continue this until only one marble remains in the box. What colour is the last marble?

Michael Pruner is the current president of the British Columbia Association of Mathematics Teachers and a full-time mathematics teacher at Windsor Secondary School in North Vancouver. He teaches using the Thinking Classroom model where students work collaboratively on tasks to develop both their mathematical competencies and their understanding of the course content.

[4] Urschel, J. (2016, December 14). The Wednesday morning math challenge: Week 14. *The Players' Tribune.* Retrieved from http://www.theplayerstribune.com/the-wednesday-morning-math-challenge-week-14/

[5] Winter 2014 problem set. (2014). *Vector, 55*(1), 47-49.

Spotlight on the Profession

In conversation with Dr. Christopher Danielson

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Dr. Christopher Danielson.

Christopher Danielson has worked with math learners of all ages—12-year-olds in his former middle school classroom, Calculus students, teachers, and young children and their families at Math On-A-Stick at the Minnesota State Fair. He designs curriculum at Desmos. He is the author of Common Core Math For Parents For Dummies, *the shapes book* Which One Doesn't Belong?, *and the forthcoming counting book* How Many? *He blogs about teaching on Overthinking My Teaching, and for parents at Talking Math with Your Kids.*

◆

First things first, thank you for taking the time for this conversation!

Besides teaching mathematics and curriculum development (at Normandale Community College and, most recently, at Desmos), one of your main interests is helping parents support their children's mathematical development, as the title of your website Talking Math with Your Kids suggests.

As you write on the website, parents know that they should read with their children every day to support their literacy development, but are typically less familiar with strategies to cultivate numeracy. Why might this be the case?

I feel I should make clear that my responses here are based on my knowledge of culture and schooling in the United States. I expect that much of this also applies in Canada, but I really don't have extensive experience to make those kinds of claims.

> "We understand that people read for pleasure, but we don't really understand that people figure things out for pleasure."

Math is tied up with school in our culture. We understand that people read for pleasure, but we don't really understand that people figure things out for pleasure. Probably our education system—and especially our mathematics education system—has a lot to do with that, but whatever the origins, this perception has consequences. If reading is pleasurable, then it's natural to curl up together on the couch with a book. If math isn't pleasurable, then we don't go looking for more of it in our daily lives.

We have clear and simple messages about supporting literacy—read out loud with children from a very young age, and surround them with books and words. But we haven't helped parents make the same kinds of connections to math. Building with blocks, doing puzzles, putting away dishes, sorting socks...these are all examples of activities that support children's math learning but we haven't pitched math this way.

So parents only know to look for things that look like school math—flash cards and their electronic equivalents.

The main strategy you offer to support children's budding numeracy skills is conversation—in particular, you recommend that parents talk (often) about math with their children "as they encounter numbers and shapes in [their] everyday lives" (Danielson, n.d.). On your website, you offer a wide variety of examples of math talks with your own children taken from everyday life, including conversations about weighing onions (2015, May 22), bedtime (2014, January 20), underpants and armholes (2013, August 17), and many more.

How do you recognize authentic opportunities for math talks, and how might parents with less experience with mathematics (i.e., without a PhD in mathematics education) develop this skill?

Yeah. That's one of the big questions at the heart of my work right now. Bedtime Math is a project that has made some headway with parents—especially math-anxious parents. They've teamed up with the University of Chicago to research the effect of non-school-based math interventions and found that introducing even small amounts of math talk into math-anxious households has meaningful positive effects on kids' achievement in elementary school math. The Bedtime Math approach to introducing this math talk is an app with simple story problems about some silly or interesting news item. Parents can open the app at bedtime (or anytime), read that day's problem, discuss with their child, and be done with it. It's a simple and elegant solution.

> "I'm greedy. I want more from kids than solving a silly story problem that someone else thought up."

But it's also a limited one from my perspective. It makes progress, but only so much. One key to its success is that it looks like school math. Everybody recognizes what it is, and so it's clear what to do with it.

I'm greedy. I want more from kids than solving a silly story problem that someone else thought up. I want the ideas children have—that *all* children have—to be the starting place for conversations and wondering and learning.

How do you help parents learn to do this? I'm figuring that out. At the outset I needed examples of the thing, and that's the origin of the Talking Math with Your Kids blog. Over time, I have started to think that a more effective strategy than writing about these conversations is putting things in the hands of teachers and students that can help to create the conversations. That's where *Which One Doesn't Belong?* and Math On-A-Stick come from.

Which One Doesn't Belong? is a shapes book that invites parents and children to have a conversation. The opening is a tutorial that makes clear everyone's ideas are valued, and that there are many ways to think about shapes. Then I turn them loose with increasingly complex sets of shapes to consider. The book closes with an invitation to look for similarities and differences in the world, and to design your own sets.

> "Having parents and kids in the same place—while the kids are using their math brains with joy—is a really powerful tool."

Math On-A-Stick is a large-scale family math event that runs all 12 days of the Minnesota State Fair each summer. There, I get to help parents notice the mathematically smart things their kids are doing, and to help them have a fun time together noticing, discussing, and playing with shapes, patterns, and numbers. Having the parents and the kids in the same place—while the kids are using their math brains with joy—is a really powerful tool. It's a limited one, though. Thousands of families come through, but we can only reach them 12 days a year, and we can only reach the families with the time and resources to attend the fair. So I'm still looking, still considering opportunities. I have ideas for parent nights that are more like Math On-A-Stick than like a traditional "How to help your kid with homework" night.

What advice do you have to offer to parents who are interested in supporting their children's numeracy development, but who harbor anxiety towards the subject—perhaps due to less-than-positive experiences in their youth?

This is important work. Often, such parents will avoid math talk out of fear that they'll harm their children. If math is about right answers, and if I'm not sure I have the right answers, then I may feel I need to steer clear of math with my kids. Plus, math anxiety is a real thing that produces a real stress response, and people tend to avoid things that make them anxious.

The University of Chicago research suggests that introducing even small amounts of math talk into such homes has a significant positive impact. I have three goals with math-anxious parents.

The first goal is to alleviate their fears that they'll harm their children by not knowing right answers. You don't harm your child when you misread a word in a book, or drop the ball she throws, or when you don't know the answer to a math question. In all of these scenarios, participating in the activity supports children's learning.

> "You don't harm your child when you misread a word in a book, or drop the ball she throws, or when you don't know the answer to a math question."

My second goal to help them notice the things they're already doing that support their children's math learning. If they're counting, or reading shapes books, or taking the kids to the grocery store, I'll make sure they know that these things are important to continue doing.

My third goal is give them one or two strategies for increasing the opportunities for learning. The simplest of these is asking *How do you know?* When a child mentions numbers, ask how she knows. Then listen to the response and compare it to how you know this. You don't need to focus on whether she is right or wrong. Instead, focus on understanding her thinking.

Presumably, as children enter their teenage years, they become less interested in conversations about weighing onions and armholes… How are the math talks you have with your son (12 years old) different from the talks you have with your daughter (9 years old)?

I have to be a lot more strategic with the 12-year-old. With him, it's more about finding ways to exploit his motivations. He loves to argue, and he relishes being right. He'll find mistakes in the world, or I'll find them and ask what he thinks. I'll make him convince me, sometimes taking up a contrary position that I don't really believe. Also he is interested in money. He loves the independence of having his own money. This leads to opportunities for making him think. When there's a job I'm paying him for, I'll make it by the hour or the pound—no flat rates. Then it's up to him to figure out what I owe him for the work, and to convince me that it's the right amount.

The 9-year-old is still a kid and finds nearly anything fun to think about or imagine. We can still talk armholes and underpants. All the examples on the blog, where the target audience is four- to ten-year-olds, still pertain to her.

Weighing onions, by the way, is still good stuff with the older child too. He likes the competitive aspect of making the best guess. Another thing we do that involves the whole family is guessing the total bill at a restaurant. I know this can contravene certain norms of polite society, but I'm telling you it's a good time and kids get good at it quickly.

You have often written that in engaging in mathematical conversations with their kids, parents shouldn't necessarily worry about whether the child can get right answers (Gahan, 2013), and should not be afraid to discuss concepts with which they are unfamiliar. As you write: "DO NOT let the idea of being wrong get in the way of your math conversations. DO NOT be afraid to play around with ideas you know little about" (Danielson, 2013, August 17).

While play and math talk may cultivate budding mathematicians' curiosity and creativity, can they—and should they—also foster precision and rigor?

Short answer: Yes!

Longer answer: Precision in play and conversation comes when there's a need for it. Often that need comes from questions that arise naturally. My daughter has been playing with stairs. She loves taking them two at a time, but is dissatisfied when there is one stair left over at the end. She has learned all of the major staircases in her life (home, school, bus, grandmother's house, etc.) and knows whether they are *even* or *uneven* (her word). We talk about this from time to time, and for a while we reached an impasse when trying to specify the number of steps that any particular staircase has. Just the other day, we focused just on that and resolved that—for us—*steps* are different from *stairs*. When she goes eight steps (in four sets of two), she counts this as 9 stairs. I only count it as 7, because I don't want to count the floors where you start and end as stairs. But we now have precise language, and the need for that precision came from needing to communicate clearly.

"Precision in play and conversation comes when there's a need for it. Often that need comes from questions that arise naturally."

But initially, the play didn't require precision and so we didn't have it. Precision without a need for it is pointless. I think this applies to school math too, and I think we all too often proceed from a place of precision and rigor without helping students to arrive at this place by routes that make sense to them. We lose a lot of potential mathematicians—and potential mathematics—that way.

Recently, you have also started offering mathematical playthings at your [Talking Math With Your Kids](http://talkingmathwithkids.squarespace.com/) online store as another way to support parents and children in math activities and conversations. These include tiling turtles, spiraling pentagons (and other tiling pentagons), pattern machines, and more [head to http://talkingmathwithkids.squarespace.com/ for details].

What do these playthings—which, on the surface, do not look particularly "mathematical"—have to do with mathematics in general, and "school" mathematics in particular?

I design things that foster kids' play with numbers, patterns, and shapes. The turtles were designed by two mathematicians—Kevin Lee, a colleague of mine at Normandale Community College, adapted them from Jos Leys' original art. Jos is a Belgian mathematician and artist. For Kevin and Jos, the turtles are artwork to be admired. I want kids to get their hands on the turtles. I don't want kids to notice what someone else has made; I want kids to make their own things.

> "Children aren't doing mathematics when they're admiring someone else's creation. They're doing mathematics when they are building relationships for themselves."

Each has its place. I love to look at and admire mathematical art. It inspires me. But children aren't doing mathematics when they're admiring someone else's creation. They are doing mathematics when they are building relationships for themselves. Figuring out how to tile the turtles is the entry level; they imagine what the turtles will look like when rotated or flipped. They notice that each turtle put into the tiling creates a space for another turtle, and so they begin to consider an unending process: infinity. Children begin to use the two colors of turtles to make patterns—alternating light and dark, or using the colors to highlight structures they notice in the tiling.

When they play with Pattern Machines, children notice and use rows and columns. Seeing and using such groups is an essential foundation for multiplication and for place value.

But as a general rule, I don't really talk a lot about the work I do in relation to school math. Mostly this is because I have critiques of school math. My daughter is a whiz with place value. She sees and understands groupings—especially groupings of ten—and is able to exploit them in clever and useful ways for thinking about things and solving problems. But it doesn't show in the ways her teachers talk about her mathematics. In school, she is supposed to say that there are 3 tens in the number 435, when she knows that there are really 43 tens there. In school, she is supposed to answer the question, *What is the value of the 4 in 435?* by writing *400* not *hundreds.* In school, she is supposed to figure out what answer the teacher wants when she asks *Which One Doesn't Belong?* So I really don't want the success of my work to be judged by the ways it prepares kids for school math.

But it does support kids' math learning in a wide range of environments, including school. My daughter may object to there being only one right answer to a *Which One Doesn't Belong?* prompt on a fourth-grade assessment, but you'd better believe she knows at least one way that each shape in the prompt is different from the others.

Lastly… why mathematics? In other words, besides the potential to increase success in academic (school) mathematics, why should parents actively work to foster their children's ability to attune themselves to the mathematics in their daily lives?

I was just talking with my 12-year-old son about this the other day. He gives me a hard time for being nerdy. As an early adolescent, it's part of his job to reject things the people around him hold dear, so I can take it. But I told him that my work really isn't about making him or his sister—or any other kid—love math. That'd be nice, but it's not the point of the work. The point of the work is for math to become a tool that they can use to do whatever they want. I know many people for whom mathematics was an obstacle that cut off possibilities. While my son gives me grief about my love of math, I see him using proportionality and probability and patterns and shapes in the thing he loves dearly—arguing (especially about politics).

"My work really isn't about making my son or his sister—or any other kid—love math. The point of the work is for math to become a tool that they can use to do whatever they want."

My daughter and I figured out together that the turntable in our microwave takes about 20 seconds to go around. She isn't tall enough to reach the back, so she always puts things in for 20 or 40 or 60 seconds. If she *needs* to do 10 seconds, she'll push the plate as far back as she can so it comes around to the front in those 10 seconds. That's math being used for making her life a little better, not for school math. More, it's empowering. She will develop an expectation that a little bit of mathematical analysis can help her do other things better, too.

Finally, mathematics as a purely intellectual endeavor is one of the beautiful cultural legacies we hand down to our children. Just like fine art or classical music doesn't need to have a practical application to be valuable, mathematics needn't be applied in order to be part of the culture we pass down.

Thank you, Dr. Danielson, for taking the time for this conversation. We look forward to your upcoming work and to continuing the discussion in the future.

Ilona Vashchyshyn

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The Cake Contest[6]

Colleen Haberern

"Wait, that's what our cake looks like?" seventh-grader Alyssa exclaimed. Her group had designed a three-tier cake that met all the requirements for the Cake Contest (see fig. 1). But when they saw the virtual model, it looked nothing like they had imagined. The cake was 36 inches tall but only 8 inches in diameter at its base. It looked like it was going to topple over! When Alyssa's group members asked if they could revise their calculations, I agreed in a heartbeat. My students were asking if they could do more mathematics! It was a math teacher's dream come true.

Fig. 1 Students were given these criteria when designing their cakes.

Design a cake that meets the following conditions:

- A serving size is 6 cubic inches.
- The whole cake must serve between 180 and 200 people.
- The cake must have at least two tiers.
- The tiers must be rectangular or cylindrical.
- Each tier must be the same height.
- The cake must be visually appealing!

Presenting Problem-Based Tasks

"Students learn best when they are presented with academically challenging work that focuses on sense making and problem solving as well as skill building" (NRC 2001, p. 335). With the adoption of the Common Core State Standards for Mathematics (CCSSM), many teachers are changing their classroom structure from teacher-directed to student-centered. When I began designing and using problem-based tasks, like the Cake Contest, I saw a drastic improvement in student engagement and problem-solving skills.

A problem-based task, which is also called a complex task, is a three-phase lesson (Van de Walle et al. 2013). First, the teacher presents a problem to the whole class in the launch, which is designed to engage students and provide a context for learning. Second, the students work, usually in small groups, to solve the task. Third, the teacher leads a whole-class discussion to help students make connections between their solutions and strengthen their understanding of the mathematical concepts.

Launching the Task

To engage my students, I presented a YouTube video showing cake designs from TLC's "The Cake Boss," with musical accompaniment from the song "Sugar, Sugar" by The Archies. As students chatted about the creative decorations, I asked them to focus on the structure of the cakes. After I had presented the task, which was to design a cake to fit a set of conditions (see fig. 1), the students realized that only the structure would matter in the

[6] Reprinted with permission from The Cake Contest, *Mathematics Teaching in the Middle School 22*(5), copyright 2016 by the National Council of the Teachers of Mathematics (NCTM). All rights reserved.

Cake Contest. Since π would need to be part of their calculations, students were allowed to round to the nearest hundredth because this approximation would have a negligible difference on the size and shape of their cakes. Technology would also play an important role because each group would create a virtual model using a 3D printer and software called Tinkercad®. Of course, every group wanted to create the "best design" because the winning group in each class would get to decorate a full-size model of their cake.

Learning the Power of Problem Solving

Each group began discussing whether to use rectangular prisms or cylinders or a combination of both. Students were familiar with these shapes, having derived and used volume formulas for both prisms and cylinders in a previous geometry unit. They sketched designs showing the number of tiers and the overall shape of the cakes. Did they know that the design they created would determine the difficulty of the mathematics to follow? If so, it did not prevent groups from being creative. They wanted to win!

> "Did they know that the design they created would determine the difficulty of the mathematics to follow? If so, it did not prevent groups from being creative. They wanted to win!"

In essence, I did not give the students a problem to solve; each group created its own. Different strategies were needed to determine the exact dimensions of the cakes required to create the virtual model. Groups used guess and check, worked backward, or made estimations, or they used a combination of strategies. The only stipulation was that they had forty minutes to figure out the dimensions.

CCSSM emphasizes real-world problem solving. In grades 5 and 6, students find the volume of cubes and other rectangular prisms, then they progress in grade 8 to additional solids, such as cylinders. Various geometry standards are incorporated, depending on each group's design. The Cake Contest also incorporates the Standards for Mathematical Practice. Students are encouraged to "make sense of problems and persevere in solving them"; in the second part of the lesson, they will use computer software to "model with mathematics" (CCSSI 2010, pp. 6–7).

Using Guess and Check

One group began the task by calculating the total volume of cake needed for 200 people by multiplying by 6. Then they sketched a cake with 3 cylindrical tiers, each with a height of 4 inches. The group used calculators and substituted values for the radius of each tier until the sum was approximately 1200 in.^{3}.

In figure 2, the radius of the top tier is much smaller than those of the other 2 tiers. Perhaps students were too focused on limiting the volume to 1200 in.^{3} and did not realize that they could add another 120 in.^{3} of cake by increasing the number of people to 220.

Making Modifications

Another group's guess-and-check strategy looked much different, although they also created a cake with 3 cylindrical tiers. Instead of focusing on the volume of the entire cake, the students divided the volume of each tier by 6 to determine the number of people that each tier would serve. After creating the bottom and middle tiers, they realized that the cake would be too small. One student suggested renaming those tiers as the middle and top, then creating a larger tier for the bottom. Since the radius of each tier was 1 inch less than the tier beneath it, the cake had the balanced look apparent in figure 3.

Fig. 2 A guess-and-check strategy was used by this group to meet the requirements of the Cake Contest.

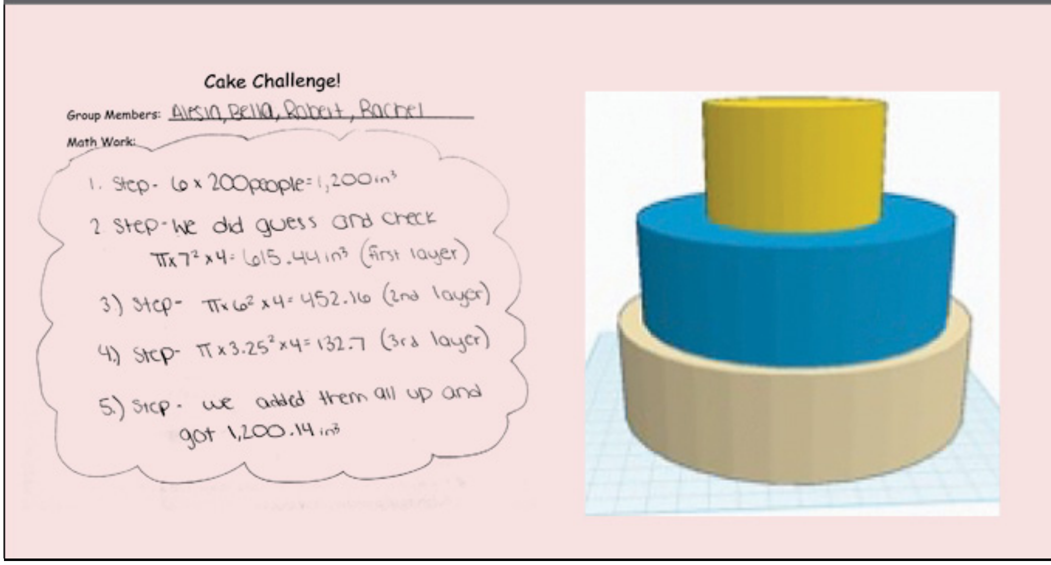
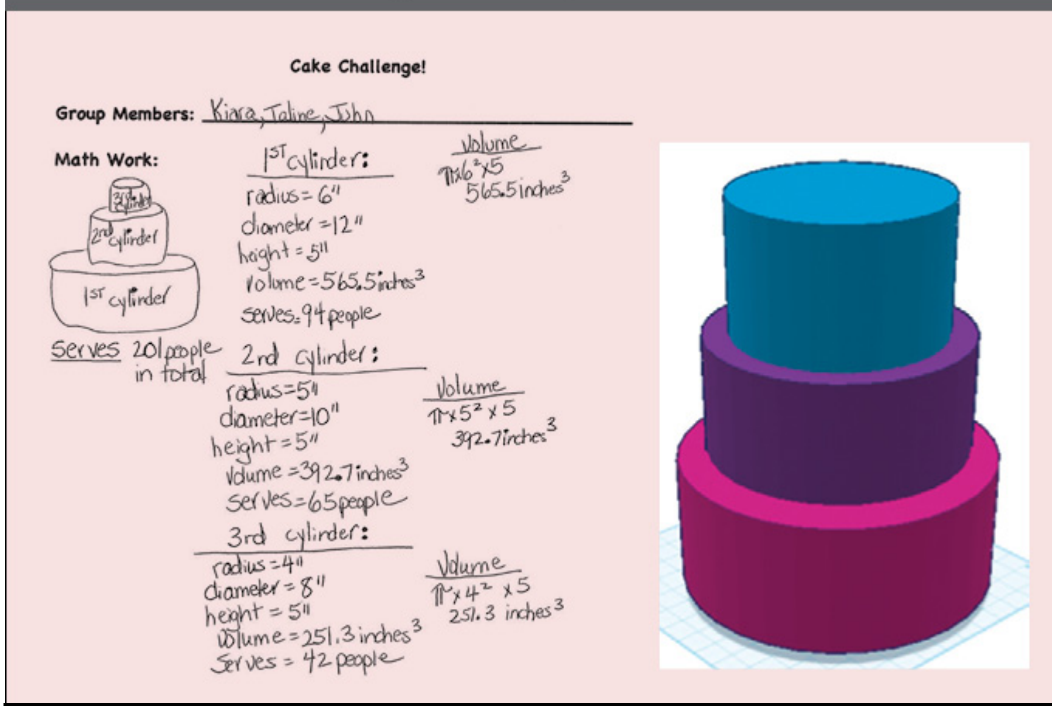


Fig. 3 When the students realized that their cake would be too small, they added a new bottom tier instead of adding a different top tier.



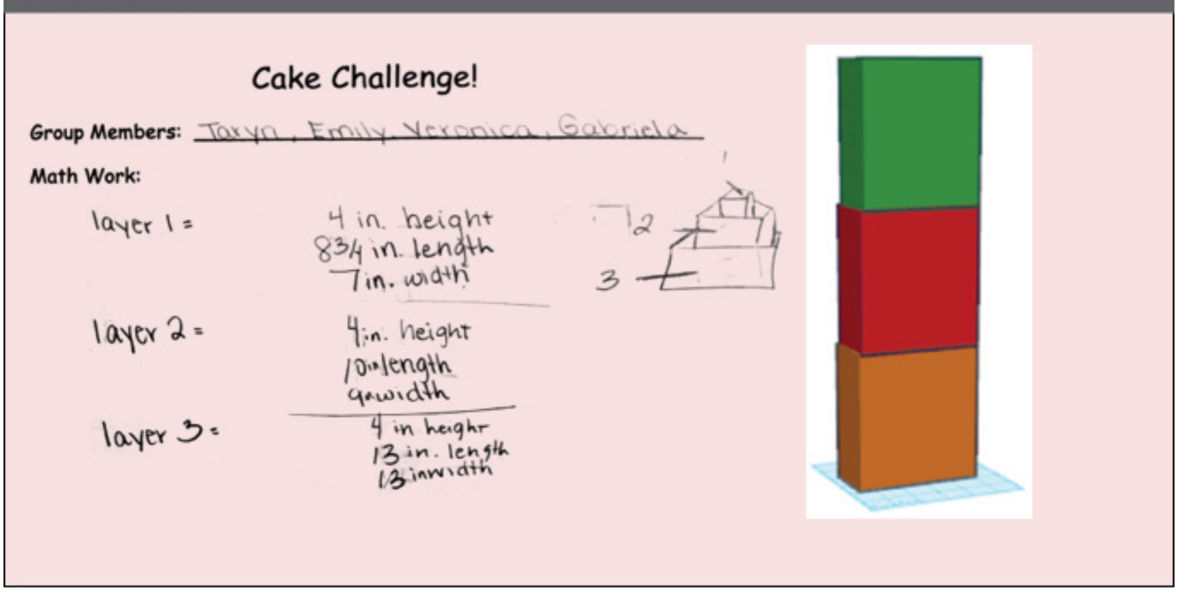
Working Backward

A third group used a working-backward strategy to create a cake made from 3 rectangular prisms. They began by multiplying 180 people by 6 cubic inches per person to get a total volume of 1080 cubic inches. Then they divided by 3 to find the average volume of each tier. The discussion below explores their thought processes:

> Taryn: So if we make one 350, one 360, and one 370, that adds to 1080. Now we just have to figure out our dimensions.
>
> Emily: Let's find the dimensions for the 360 tier first because that seems like the easiest to do. . . . If we divided 360 by 3, we get 120, but there has got to be an easier way. Maybe something by 10 . . . 9 × 4 × 10 equals 360. Right?
>
> Gabriella: Yeah.
>
> Emily: I say we should make 10 the height because it is a number they all have in common.
> Taryn: How?
>
> Emily: They all end in 0, so they are all multiples of 10.

The group questioned whether or not a 30 inch tall cake was acceptable. They concluded that it was reasonable because the cakes on the "Cake Boss" videos are quite large. Then they went back to work calculating the dimensions of the other tiers. Their work appears in figure 4.

Fig. 4 It was surmised that using rectangular prisms would make the work of cake design less challenging. This treatment led to some in-depth conversations about mathematics.



> Taryn: The top one is 35, so we could do 7 and 5.

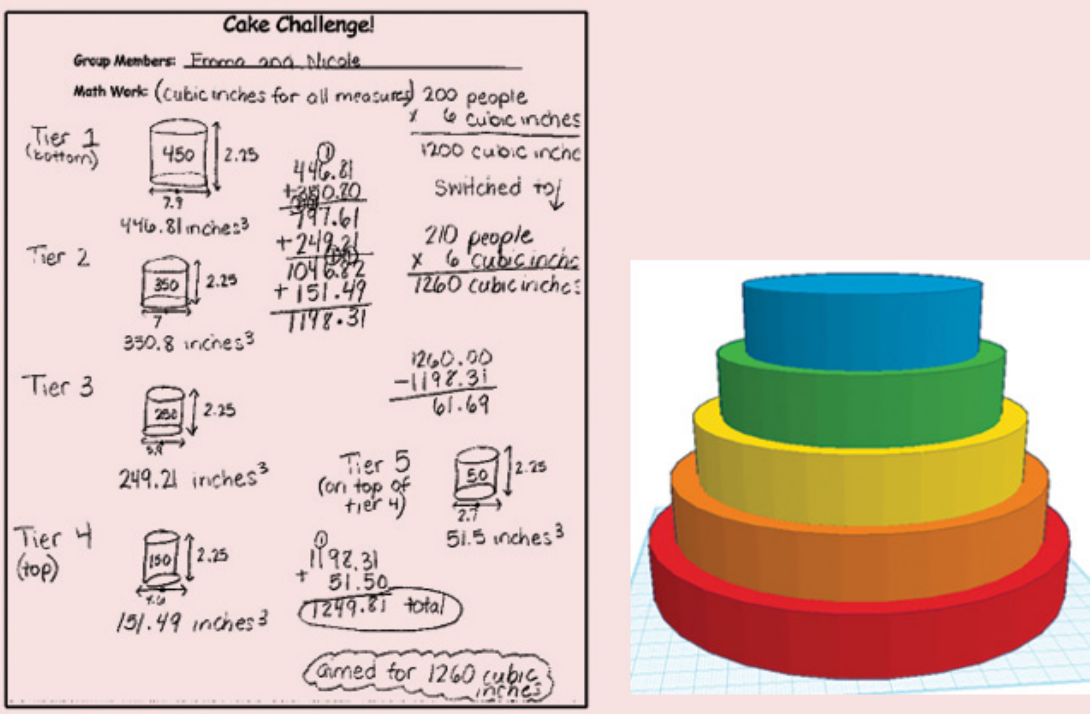
Emily: Hey, that would be hanging off the cake! Is there anything else that we could multiply by? Or we can have a fraction of an inch on the side.

Emily realized that they had to consider how the tiers would fit together to create the cake. She knew that the length and width of each tier must be less than or equal to the corresponding dimensions beneath it. Since there are no other integral factors of 35 between 4 and 9, the group decided to make the width of each tier constant, then use division to calculate the missing lengths.

Creating a Need for Algebra

Working backward looked different for the group that created a cake with 5 cylindrical tiers (see fig. 5). They originally wanted 4 tiers and began by calculating the total volume of cake required for 200 people then divided by 4. They started with 450 cubic inches on the bottom tier and decreased the volume of each tier by 100 in.[3]. This constant change in size created a roughly constant slope along the profile of the cake.

Fig. 5 A working-backward strategy was used to create a cake with 5 cylindrical tiers.



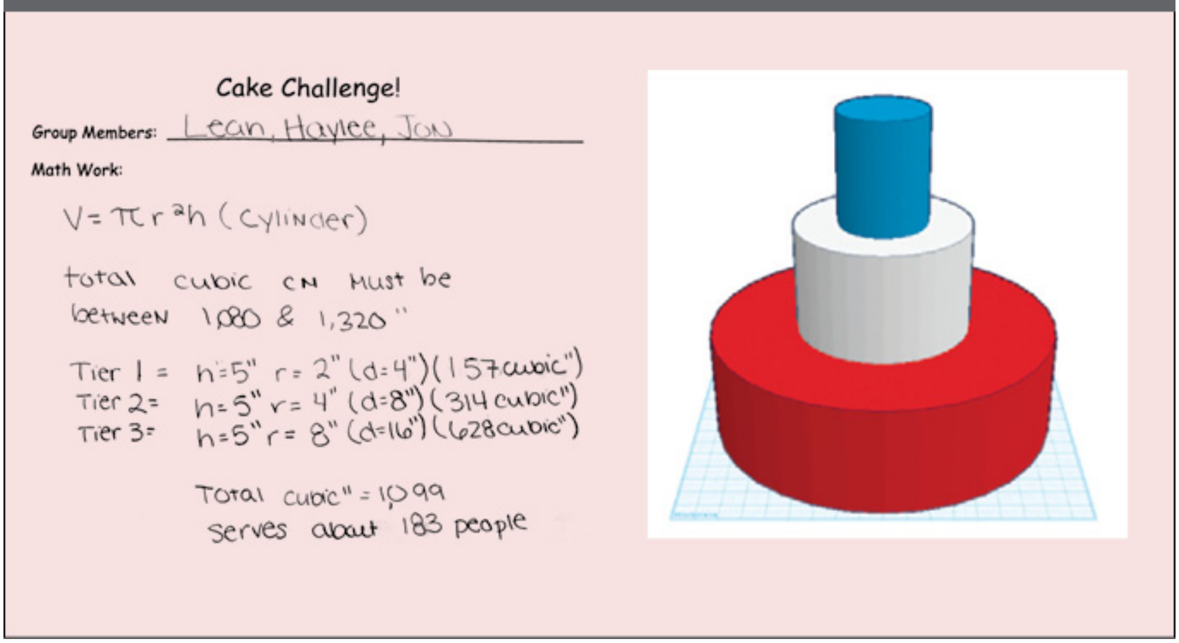
Once students determined the volume of each tier, the group decided that the height of each tier would be 2.25 inches. At this point, the group worked backward to calculate the radius of each tier. The students knew that $\pi r^2 \times 2.25 = 450$, so they used calculators to divide 450 by 2.25π, then took the square root to find the radius of the base. Although we had previously covered square roots and cube roots, I thought it was impressive for seventh graders to devise this calculation, especially considering that formal algebra had not been covered at that point in the school year. They used the same process to find the radius of the other 3 tiers.

The students checked their work by substituting the radius and height into the formula for the volume of a cylinder. They added to find the total volume of the cake and realized that they could add another tier. That turned out to have been a smart choice because this group was voted “best design” in their class.

Using Estimation
As I checked in on another group, I could see that they used “ballpark estimation” to find the minimum and maximum volume of cake. Because all their dimensions were whole numbers, I assumed that they had used guess and check. I asked the group to tell me about the process.

“I just thought about how big a real cake would be,” explained Leah, holding out her hands to approximate the size. The other group members showed me how they used rulers to measure the height of the tier that Leah was representing with her hands. They had measured all three tiers, and the total volume fell within their range on the first try. Apparently, Leah’s mother was a chef and often made wedding cakes at home. (See fig. 6.)

Fig. 6 Group members estimated the dimensions of a real cake, aided by a pastry chef’s daughter.



Employing Multiple Math Concepts
The other winning group used both rectangular prisms and cylinders in their design. They chose to use the maximum number of people, so that they would have the greatest volume of cake, which was 1320 $in.^3$. A student suggested that the 4 tiers, shown in figure 7, should contain 10 percent, 20 percent, 30 percent, and 40 percent, respectively, of the total volume

of cake to create a smooth look. They multiplied the decimal form of each percentage by 1320 in.^{3} to calculate the volume of each tier.

The students thought it would be easier to find the dimensions of the rectangular prisms versus the cylinders, but they were unsure how to proceed. The group wanted the top and bottom faces of the prisms to be square, which added to the difficulty level. After a short discussion, one student suggested taking the cube root of the volume of both rectangular prisms, giving side lengths of approximately 5.09 inches for the top tier and 7.34 inches for the third tier.

At this point, the group asked me for help with the cylinders. I reminded them that the Cake Contest rules stated that all tiers had to be the same height. Not only did my comment help the students realize that the cubes they created were incorrect, it also gave them a starting point for the cylinders.

The students chose a height of 3 inches for each tier. They divided the volume of the large square tier by 3 to get 44, which was the area of the base of the rectangular prism. Then they took the square root of 44 to find the length and width. They repeated the process for the top tier. To find the dimensions of the cylinders, they divided the volume by the height to get the area of the circular faces. Next, they divided by π to find the radius squared. By taking the square root of that number, they were able to determine the length of the radius.

After working through obstacles to find the dimensions, the group felt a sense of accomplishment. However, they would face yet another challenge when they created a virtual model of their cake.

Virtual Models

On the second day of the Cake Contest, students used computer-aided design (CAD) software, called Tinkercad, to create virtual models. The students had no prior experience using this software but quickly learned after viewing Tinkercad's video tutorial. When "modeling

Fig. 7 This group used both cylinders and "square" rectangular prisms in their design. (See their virtual image in **fig. 10.**)

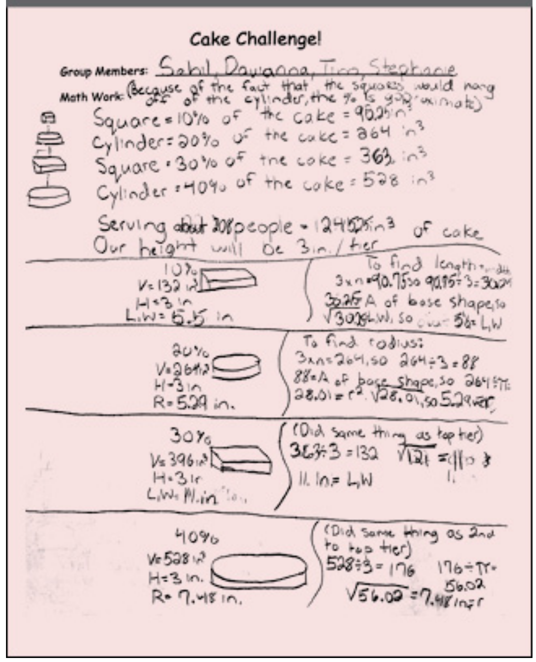
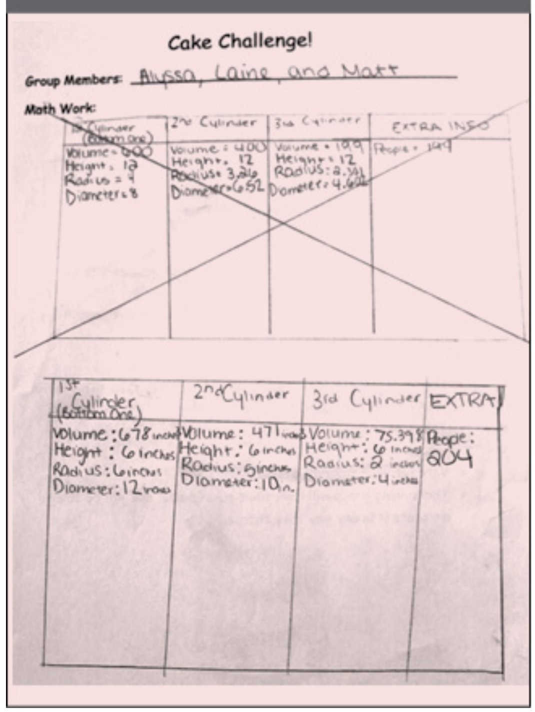


Fig. 8 Alyssa's group made last-minute revisions to their cake.



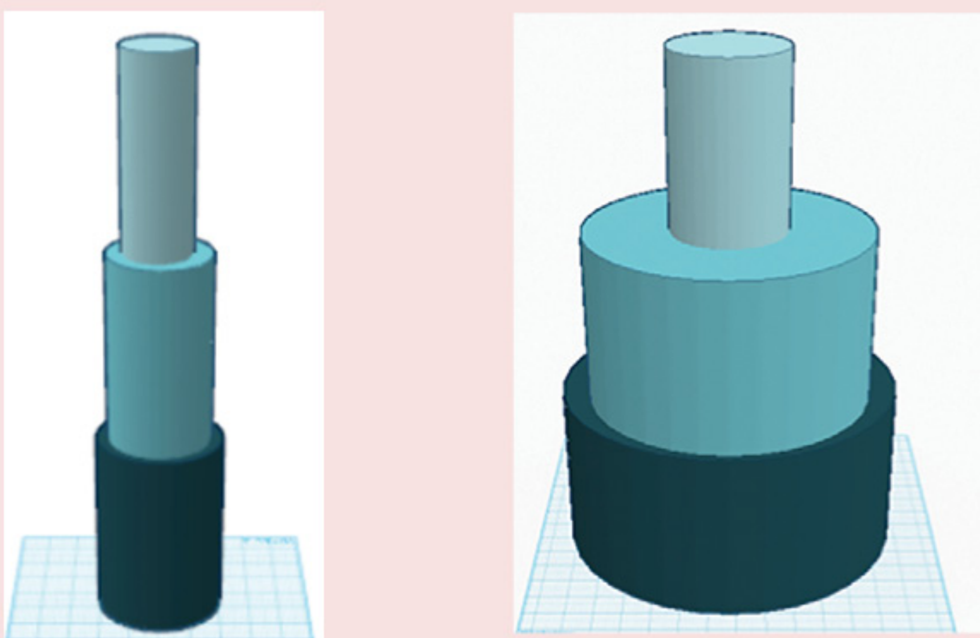
with mathematics," students often draw diagrams as they imagine them to look. Using the Tinkercad software, students were able to see what their cakes would actually look like. It became a powerful learning tool.

Learning from Mistakes

Two groups did not like what the Tinkercad software generated, so they asked me if they could make changes. Alyssa's group originally designed tiers with a height of 12 inches. They gradually decreased the volume of each tier from 600 to 400 to 199 in^3. They divided by the height, divided by π, and took the square root to calculate the radius of each tier. The virtual image was much taller and narrower than they expected, so they reduced the height of each tier from 12 inches to 6 inches. In turn, they increased the radius of each tier to keep the volume within the given range, as seen in figure 8.

Perhaps because they were running short on time in the contest, the group did not continue to use the same strategy. Instead they selected a radius for each tier and calculated its volume. So as to keep the total number of people served close to 200, they made the top tier much smaller. This last-minute correction made the cake more stable but changed the appearance to a less even look, as shown in figure 9.

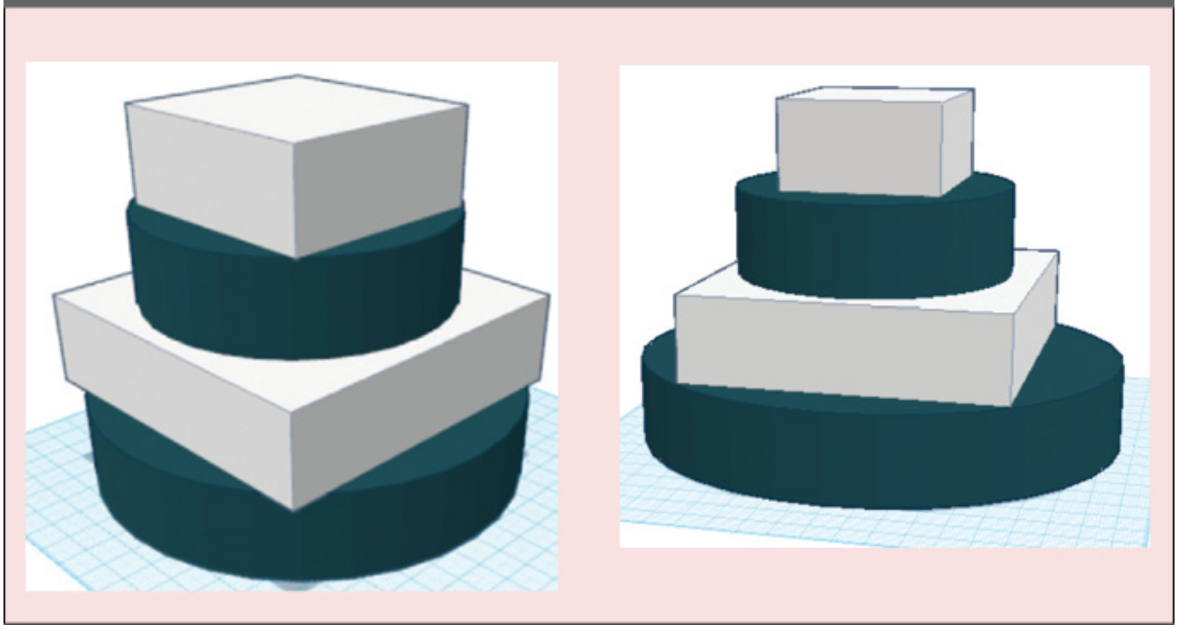
Fig. 9 Reducing the height of each tier made the original cake (a) more realistic (b).



Overhanging Tiers

When the group that alternated cylindrical and "square" tiers created their virtual cake image on Tinkercad, they noticed that the corners of the "square" tiers were hanging over the edges of the cylinders (see fig. 10). They had planned carefully to ensure that the length of each square was less than the diameter of the cylinder beneath it. They did not understand the problem, so they called me over for help.

Fig. 10 In this design, the corners of the "square" tiers were hanging over the edges of the cylinders. This group corrected their error, and their cake was voted "best design."



I told students that they had not taken into account the fact that the diameter is the widest part of a circle but that the corners of the "square" tiers were overlapping a narrower part of the circle. One student asked indignantly, "How were we supposed to know that?" To answer, I explained that virtual models serve a purpose. If there is an error in the design, it could be corrected. He remembered that they had made a cake with the greatest volume, so they reduced the length and width of the rectangular prisms to make them fit on the cylinders.

Differentiating Instruction

This problem-based task was completed by seventh-grade students working in groups of three or four. The student work shown was selected from two advanced classes with approximately 20 students in each class. However, this task can be differentiated and used with other grade levels or abilities; see the sidebar on the next page.

When presented with an open-ended problem like the Cake Contest, students are able to choose an appropriate difficulty level. Even though I required a minimum of two tiers, all groups made a cake with three or more tiers. I heard one student suggest using two tiers but her group convinced her that adding an extra one could help them win. In addition, eight out of eleven groups included cylinders, even though the formula is more difficult than the formula for rectangular prisms. I structured the task so that the groups could not produce a cake design that they did not have the ability to determine dimensions for.

Finding Connections

In the past, I taught mathematical concepts in isolation by focusing on one objective at a time. Having been taught this way as a student, I only recently recognized how many connections there are among mathematical concepts. Now one of my goals is to help my students make those connections by combining various areas of mathematics in one problem-based task. "Problem-based learning (PBL) works well with all students, making its strategies ideal for heterogeneous classrooms where students with mixed abilities can pool their talents collaboratively to invent a solution. . . . By allowing children to direct their own activities and by giving them greater responsibilities, teachers show them how to challenge themselves and learn on their own" (Delisle 1997, p. 7).

The Common Core State Standards Writing Team stated, "Problems involving areas and volumes extend previous work and provide a context for developing and using equations" (2011, p. 18). Although this was a geometry task, my students used measurement, estimation, percentages, square roots, and algebra. They are beginning to understand the need for various types of mathematics. Tasks like the Cake Contest help reinforce concepts previously learned and provide a real purpose for using them.

Suggestions for Differentiation

These suggestions for making the task more or less challenging for various ability levels will help this lesson become accessible for all students.

1. Vary the number of people.
2. Allow a wider (or narrower) range.
3. Assign a height for each tier.
4. Change the number of tiers required.
5. Assign the types of solids to be used.

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Talking Points

Heidi Neufeld

I think it was my internship partner teacher who first introduced to me a 'talking points' structure:

> A statement is projected on the board, students are asked if they agree or disagree and why, and are given 3 minutes to silently write out their own response. Then, they have 3 minutes to discuss their response with a peer next to them, and afterwards, we have 3 minutes for a classroom discussion… which rarely ends after 3 minutes.

Since my internship, I've heard of other math teachers using their own version of talking points, including Chris Luz (@PIspeak) and Elizabeth Statmore (@cheesemonkeySF; see her blog post entitled "Talking Points Activity – Cultivating exploratory talk through a growth mindset activity"). The structure seems to have been developed by Lyn Dawes for use across the elementary curriculum in her book, *Talking Points* (2012). (Although the book is written for teachers at the elementary level, I've found certain elementary resources helpful in reflecting on my teacher moves at the secondary level as well. *Intentional Talk* by Elham Kazemi and Allison Hintz, which focuses on mathematical discussions in the elementary context, is another good example.)

This year, I've started incorporating talking points into my Foundations 10 class a few times every unit.

Of course, a good talking point requires a good statement to start with. For me, this means a statement that is clear enough for everyone to quickly understand what is being said, but ambiguous enough for multiple perspectives to emerge. For example, during our unit on linear functions, I used the statement "Any straight line is the graph of a linear function," which brought up all the 'weird' cases of linear relations (vertical and horizontal lines, including $y = 0x + 0$), and got students thinking again about the difference between a relation and a function. My collection of talking points (which I've found on blogs, in the textbook, and even in the Saskatchewan curriculum itself) is currently small, but now as I look for task ideas for upcoming units, my eyes are peeled for statements with these qualities.

> "During one class in particular, the students blew me away with their ability to articulate their reasoning, think critically about the arguments of their peers, and ask great "what if" questions."

The talking points have almost always led to interesting discussions, but during one class in particular, the students blew me away with their ability to articulate their reasoning, think critically about the arguments of their peers, and ask great "what if" questions. The context was systems of linear relations.

To start off the unit on systems of linear relations, we spent some time practicing graphing lines using slope and y-intercepts, as well as using x and y intercepts (skills that the students had started developing in the last unit, but needed to master over the course of this unit). I explained that when we draw two lines on the same graph, we call it a system of linear equations, and that the point of intersection is called the solution to the system because it's on both lines, and therefore satisfies both equations. Then, we moved into an awesome activity by John Orr where we developed systems of linear equations based on data collected during an in-class "trashketball" competition. Students calculated their average number of shots made per minute, we graphed the results, and predicted who would always win if we pitted certain teams against one another. Then, we made things "more interesting" by giving the team with the smaller average an advantage of a few balls already in the bucket at time zero, which added a *y*-intercept to graph, and created a system with a solution corresponding to the time when the teams would tie. Of course, we had to actually try it out and see if the competition would really end in a tie. Do check out the full description on John's blog. It's a great task that gives contextual meaning to the solution of a system and allows for awesome cross-curricular conversations (students were surprised that their solutions didn't always line up with reality, and we talked about the differences between doing science and doing math).

I was now ready to introduce the properties of different types of linear systems. As the students came into class one afternoon, one of them made my day by asking, "Can we argue again today? That's fun." To which I happily responded, "I hope you do!"

"Can we argue again today? That's fun."

On this day, I ask the students to take out a piece of paper and get ready to write their response to a new talking point:

"Two lines always intersect at one point."

The students get to work writing out their responses. I had recently heard about Steve Leinwand using the framework, "convince yourself, convince a friend, convince a skeptic," which originally comes from Jo Boaler and Cathy Humphreys' book, *Connecting Mathematical Ideas* (2005). And so, as the students start writing, I remind them that their argument needs to be water-tight. Now is the time to fully convince themselves and their peers of their position, using whatever examples or counter-examples might strengthen their argument.

"I remind them that their argument needs to be water-tight. Now is the time to fully convince themselves and their peers of their position."

After a few minutes of silence, I ask them to pair up with someone sitting next to them and share their responses. I circulate and prompt students to think further about their responses. If they had the same response as their partner, I ask the pair to think about and write down how they think other students may have responded. If they have different responses, I ask them to try to convince each other of their position. While circulating, I'm looking to see who had common, unique, ambiguous, or well-articulated responses and choose particular students to ask if they'd be willing to share during the whole-group discussion.

When the conversations die down, I ask a student who had a common response to share their reasoning, and then ask if anyone else had a similar line of reasoning and if they would share it. After asking if anyone has a different response, things seem to get crazy, and I go with the flow as students share their thinking, respond to one another, ask questions, drop their jaws, and make "my head just exploded" gestures. My role becomes less that of "I'm a teacher who wants you to make this specific curricular connection" and more "I'm a moderator who makes sure students get a chance to finish their statement before someone jumps on top of them with a rebuttal." I also try to highlight the questions that are posed, write them on the board, and ask students to go back to their pairs to discuss what they think about the question before continuing. I hope to give students a chance to think before the conversation whizzes past them.

This time around, the common response was, “No, they don’t always intersect, e.g. parallel lines.” Other students used the counter example of a line intersecting a curve more than once, to which some students replied, “That’s not a line!” Others figured that if a line started somewhere and kept going, and another line started near it but kept going in the other direction, they would never intersect, to which, again, some students replied, “That’s not a line!” I love when the argument boils down to a definition: what better way for students to really experience the practice of doing mathematics—they argue about something and realize that it would be much easier if they all interpreted a particular word in the same way, bringing about the *need* for a definition. Some said a line doesn’t have thickness, and others said it could, and would then be a “thick line,” at which point I drew a one and asked if it was a line or a rectangle.

> “I love when the argument boils down to a definition: what better way for students to really experience the practice of doing mathematics?”

Eventually, the students started pleading with me to go ask the math teacher next door to come and tell us what a line is, or to call their Math 9 Enriched teacher and ask him. I couldn’t help but wonder what professional mathematicians might do in this situation: Do they have access to an “authority figure” they can bring in to clear up controversy? I told the students, “You know that if I call your Enriched teacher, he’s going to just ask you what you think and why,” at which point they all slouched with the realization that I was right.

At one point, a student asked, “If you take two parallel lines and put them on top of each other, then is that one line?” Others chimed in: “Or would that be a thicker line?” At this point, I played ‘teacher’ and asked how many solutions there would be to this system of linear equations. Later, the same student asked, “If a line has no thickness, then couldn’t there be an infinite amount of lines?” Some students dismissed this comment as ridiculous, but I couldn’t wait to ask about equations, and how many different equations could be used to draw lines that all fall on top of each other.

The conversation went on for a full 40 minutes.

We talked about lines, curves, line segments, rays, parallel lines, slope, systems with no solution, systems with one solution, systems with infinitely many solutions, infinity itself, how equations represent lines, and even briefly mentioned non-Euclidean geometry (parallel lines never intersect? Well, actually…).

> “Students gave examples, argued their position, changed their mind, asked questions, explored possibilities, and engaged in the process of doing mathematics.”

Students gave examples, counter-examples, argued their position, changed their mind, asked questions, explored possibilities, and engaged in the process of doing mathematics.

Sure, I could have taken far less time by simply telling them, “Intersecting lines have one solution, coincident lines have infinitely many solutions, and parallel lines have no solution,” but this was an opportunity to allow the students to operate mathematically under intrinsic motivation, and I’m so glad I didn’t pass it up.

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Heidi Neufeld is a mathematics teacher within the Saskatoon Public School Division. Throughout her internship and her time at the College of Education, as well as in her current work with high school students, she has grown in her excitement to be a teacher who values deep conceptual understanding and positive relationships in her mathematics classroom. Heidi loves learning and collaboration, is a proud aunty, and tweets about math education at @HeidiLNeufeld.

Matching Tests [7]

Jehu Peters

An interesting coincidence happened today in class. My co-teacher was handing the students back their tests, wanting each student to mark another's test. However, she accidentally handed a student back her own test. She casually said to me, "weird, what are the chances of that?" Her question implied that this was a rare event. The answer is quite the opposite.

"My co-teacher accidentally handed a student back her own test. 'Weird, what are the chances of that?' Her question implied that this was a rare event."

My co-teacher had a 63%, $\frac{e-1}{e}$ to be precise, chance of handing at least one student back his or her own test. If you are wondering, *e* stands for a special irrational number; e = 2.71828… This percentage is not very good. It means that, if you randomly distribute a batch of tests to students to mark, about two thirds of the time at least one student will end up with his or her own test!

How in the world did I come up with that answer? Let me back up a bit. Suppose we had 3 students and we wanted to give them back their tests to mark so that no student had his or her own test. What options do we have? One option is:

Student A: Test B
Student B: Test C
Student C: Test A

Another option is:

Student A: Test C
Student B: Test A
Student C: Test B

These are the only two possibilities. Hence, we have 2 scenarios where we will hand back the tests with no matches. How many options do we have in total? You can check for yourself that there are 6 possible test arrangements.

Thus, we have a 2/6 = 33.3% chance that everyone has a different test. In other words, a 66.6% chance that at least one student receives his or her own test to mark (which would *not* be good for our academic standards).

What if we had 4 students? If you work it out, you will find that there are 9 options where the students have different tests. For example:

Student A: Test B
Student B: Test C

[7] A prior version of this article was published on October 5, 2015 on Jehu's blog at https://themathbehindthemagic.wordpress.com/2015/10/05/matching-tests/. Reprinted with permission.

Student C: Test D
Student D: Test A

Again, basic counting principles dictate that there will be $4! = 24$ possible test arrangements. Hence, there is a $9/24 = 38\%$ chance each student will receive another student's test to mark. We could continue on like this, painstakingly grinding out the probabilities for each situation until we reached a class size of 25. However, there is a pattern we can exploit. I created the following table using a Python script:

Number of students	Fraction of proper distributions
3	$2/6 = 33\%$
4	$9/24 = 38\%$
5	$44/120 = 37\%$
6	$265/720 = 37\%$
7	$1854/5040 = 37\%$
8	$14833/40320 = 37\%$

Do you see the pattern? Once the class size gets to 5, the percentage becomes stable. We could predict that a class of 25 students will have roughly the same chance of each student receiving someone else's test, 37%. Indeed, one can prove mathematically that this percentage will be approximately true for any size of class, large or small. For now, this is left as an exercise for the reader; I will share my own proof in the next issue of *The Variable.*

Thus, my advice to teachers is as follows: Be careful when distributing tests to students to mark. You are playing a game with a 37% chance of winning. And as any gambler knows, those are not good odds.

Jehu Peters is a recent graduate from the Faculty of Education at the University of Winnipeg. He teaches Mathematics at Maples and Garden City Collegiate. Jehu is interested in problem solving, algebra, calculus, and enjoys the connections between mathematics and science. He is currently studying for his Masters in Mathematics for Teachers from the University of Waterloo.

Reflections *is a monthly column for teachers, by teachers on topics of interest to mathematics educators: reflections on classroom experiences, professional development opportunities, resource reviews, and more.*

Extreme Math Challenge 2017

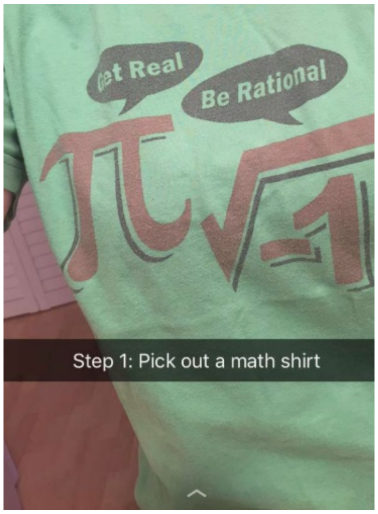
Amanda Culver

Between Walter Murray Collegiate and Centennial Collegiate, we host the Extreme Math Challenge and the Extreme Math Camp every year. On Saturday, March 18th, we launched our first full-day Extreme Math Challenge. In the past, this event was held after school. However, we always felt that our schedule was rushed and would have liked to have more time with students. And thus, the weekend Extreme Math Challenge was born.

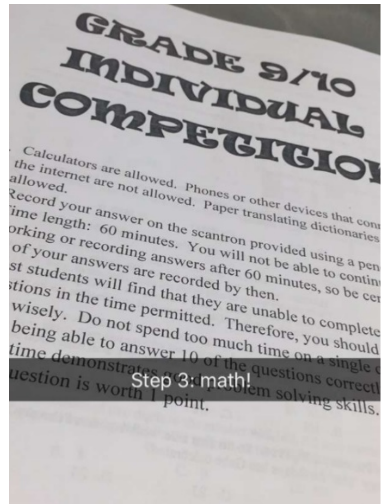
The brains behind this event (Cam Milner and Aditi Garg) planned a fun day of math for students from Grades 7 to 10. Students pre-selected teams for the team round of the challenge and also participated in an individual round of questions. A fun addition this year was the relay. To encourage collaboration and get students working with peers from different schools, students were randomly split into smaller teams upon their arrival in the gym. There, they had to complete a variety of activities to get from one end of the gym to the other in order to answer a math question before heading back to their team. Students spent about an hour being active while also completing some quick questions.

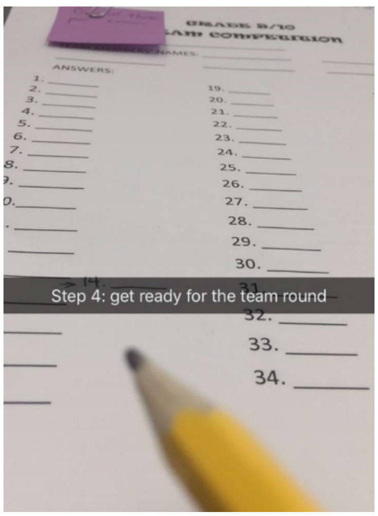
The day was interspersed with brain teasers, lunch, and snacks. Cookies were provided—but not just any cookies: they were decorated with circle properties! Students also had the opportunity to decorate their own cookies. Of course, we like to recognize students for coming out on a Saturday to do math, so the day concluded with some individual, team, and relay round prizes (including calculators, books, and candy).

Here's what my day looked like:

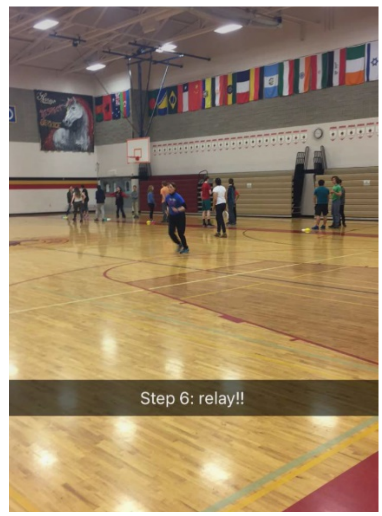

Be Rational
Step 1: Pick out a math shirt


Step 2: load up on math cookies and Timmies


GRADE 9/10
INDIVIDUAL
Step 3: math!

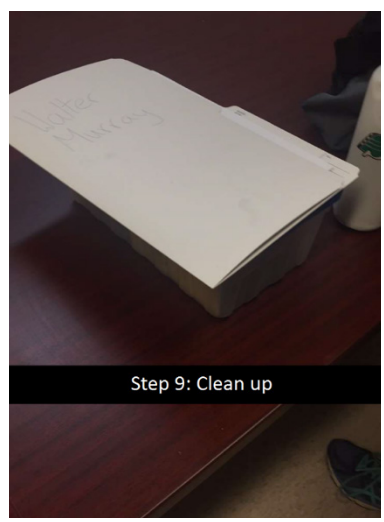

Step 4: get ready for the team round


Step 5: lunch! (There was pizza)


Step 6: relay!!


Step 7: basketball relays with math 👍


Step 8: Prizes


Step 9: Clean up

Needless to say, everyone left feeling well-challenged, which means that our job was accomplished. New friendships were made, and we anticipate seeing many of these faces this summer at our Extreme Math Camp.

If you weren't able to make it to Extreme Math Challenge this year, don't worry—you and your students can still join in on the fun by exploring a few of the problems from the competition (below). If you find interesting solutions, share them by submitting them to *The Variable*!

1. If $\frac{N}{30}$ is a simplified fraction (that is, reduced to lowest terms) and is between 0 and 2, how many values are possible for N?

2. When the hands of a clock indicate the time is 9:20, then what is the measure of the obtuse angle formed by the hands, in degrees?

3. Howard goes out for a run every 3 days, goes to the store every 5 days, and watches a movie every 7 days. If he did all 3 things on September 1, then on what date (give month and day) does he next do all three things on the same day?

4. How many ways can you make 8 boy-girl pairs with 8 boys and 8 girls?

What are the last two digits of 11^{2016} and 7^{2016}?

Amanda Culver has been a French and mathematics secondary teacher within the province of Saskatchewan for four years. She aims to make her classroom a safe and supportive space to be and to learn mathematics. Amanda's closet is full of math t-shirts, and she got a "pi" tattoo on Ultimate Pi Day. Needless to say, she loves math!

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description. If you know about an event that should be on our list, please contact us at thevariable@smts.ca.

Within Saskatchewan

Workshops

Using Tasks in Middle Years Mathematics
August 9, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Using tasks in a middle years mathematics classroom can provide rich opportunities for differentiated learning and authentic assessment.

How do we choose tasks that meet both curricular outcomes and student needs? Tasks allow students to enter mathematics where they are at and extend their learning. In this workshop we will look at a variety of resources for finding good middle years tasks. We will also reflect and discuss what planning and teaching moves can assist in maximizing student learning through mathematics tasks.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/using-tasks-middle-years-mathematics-0

Number Talks and Beyond: Building Communities Through Classroom Conversation
August 10, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Classroom discussion is a powerful tool for supporting student communication, sense making and mathematical understanding. Curating productive math talk communities requires teachers to plan for and recognize opportunities in the live action of teaching. Come experience a variety of classroom numeracy routines including number talks, counting circles, quick images and more. Take math conversations to the next level by strengthening your skills as a facilitator of classroom discourse and student thinking.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/number-talks-and-beyond-building

Conferences

Saskatchewan Understands Math (SUM) Conference

October 23-34, Saskatoon, SK
Presented by the Saskatchewan Mathematics Teachers' Society (SMTS), the Saskatchewan Educational Leadership Unit (SELU), and the Saskatchewan Professional Development Unit (SPDU)

This year, the Saskatchewan Mathematics Teachers' Society, the Saskatchewan Educational Leadership Unit and the Saskatchewan Professional Development Unit are partnering to co-ordinate a province-wide conference to explore and exchange ideas and practices about the teaching and learning of mathematics. The Saskatchewan Understands Math (SUM) conference is for mathematics educators teaching in Grades K-12 and all levels of educational leadership who support curriculum, instruction, number sense, problem-solving, culturally responsive teaching, and technology integration, and will bring together international and local facilitators to work in meaningful ways with participants in a variety of formats. This year, SUM is featuring keynote speakers Steve Leinwand of the American Institutes for Research and Lisa Lunney-Borden of St. Francis Xavier University. See the poster on page 4, and head to our website for more information.

Beyond Saskatchewan

MCATA Fall Conference 2017: A Prime Year for Mathematics
October 20-21, Enoch, AB
Presented by the Mathematics Council of the Alberta Teachers' Association

Join the Mathematics Council of the Albeta Teachers' Association in celebrating their annual fall conference in Enoch, Alberta. This year's keynote speakers are Michael Pruner, a high school mathematics teacher with a Thinking Classroom in North Vancouver and president of the BC Association of Mathematics Teachers, and Sunil Singh, author of the book, *Pi of Life: The Hidden Happiness of Mathematics* and a self- proclaimed Mathematical Jester who is transforming the way mathematics is revealed and discussed all over North America.

See http://www.mathteachers.ab.ca/information-and-registration.html

NCTM Annual Meeting and Exposition
April 25-28, 2018, Washington, DC
Presented by the National Council of Teachers of Mathematics

Join more than 9,000 of your mathematics education peers at the premier math education event of the year! NCTM's Annual Meeting & Exposition is a great opportunity to expand both your local and national networks and can help you find the information you need to help prepare your pre-K–Grade 12 students for college and career success. Classroom teachers, administrators, math coaches, supervisors, college professors, and preservice teachers can all benefit from the sessions and learning at this event. Improve your knowledge and skills with high-quality professional development and hands on activities; gain insights by connecting and sharing with like-minded educators; collect free activities that will keep students engaged and excited to learn; and learn from industry leaders and test the latest educational resources.

See http://www.nctm.org/Conferences-and-Professional-Development/Annual-Meeting-and-Exposition/

Online Workshops

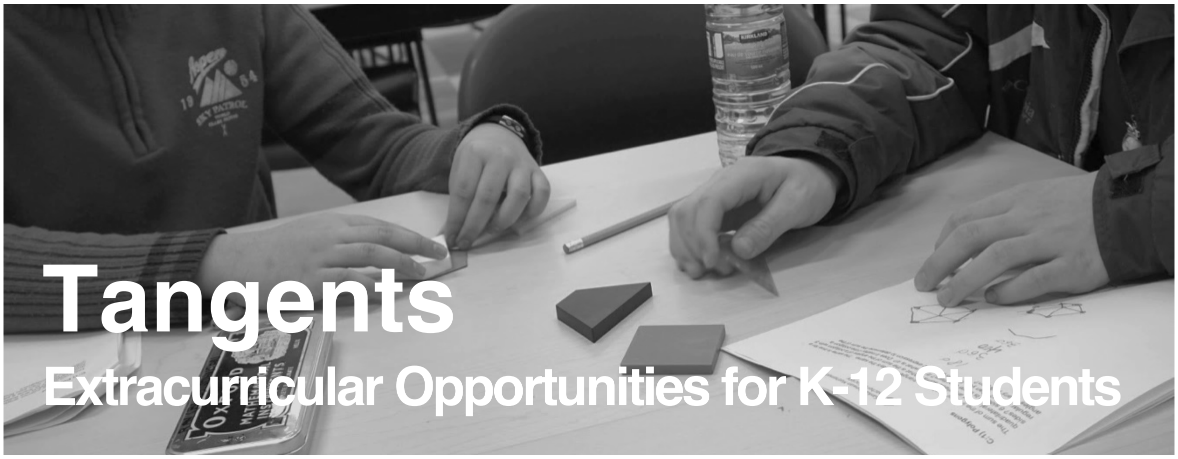
Education Week Math Webinars
Presented by Education Week

Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling and Differentiation.

See http://www.edweek.org/ew/marketplace/webinars/webinars.html

Did you know that the Saskatchewan Mathematics Teachers' Society is a **National Council of Teachers of Mathematics Affiliate**? When registering for an NCTM membership, be sure to support the SMTS by noting your affiliation during registration.

NCTM | AFFILIATE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS



Tangents

Extracurricular Opportunities for K-12 Students

This column highlights local and national extracurricular opportunities for K-12 students interested in mathematics, including collaborative, individual, online, and in-person challenges, contests, and camps. For dates, registration procedures and applications, and other information about the contests listed, please head to the contest websites, included below the descriptions. If we have missed an event that should be on our list, please contact us at thevariable@smts.ca.

If you are looking for contests available in other provinces, head to the Canadian Mathematical Society website (cms.math.ca/Competitions/othercanadian). The CMS also maintains a list of resources for students who are looking to build their problem-solving skills and succeed in competitions: see cms.math.ca/Competitions/problemsolving.

Caribou Mathematics Competition
May 2017

The Caribou Mathematics Competition is a worldwide online contest that is held six times throughout the school year. Each of these days, five contests are offered, one for each of the grade levels 3/4, 5/6, 7/8, 9/10 and 11/12 and each one in English, French and Persian. Available in English, French, and Persian.

See https://cariboutests.com/

Gauss Mathematics Contests
May 2017
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Gauss Contests are an opportunity for students to have fun and to develop their mathematical problem-solving ability. For all students in Grades 7 and 8 and interested students from lower grades.

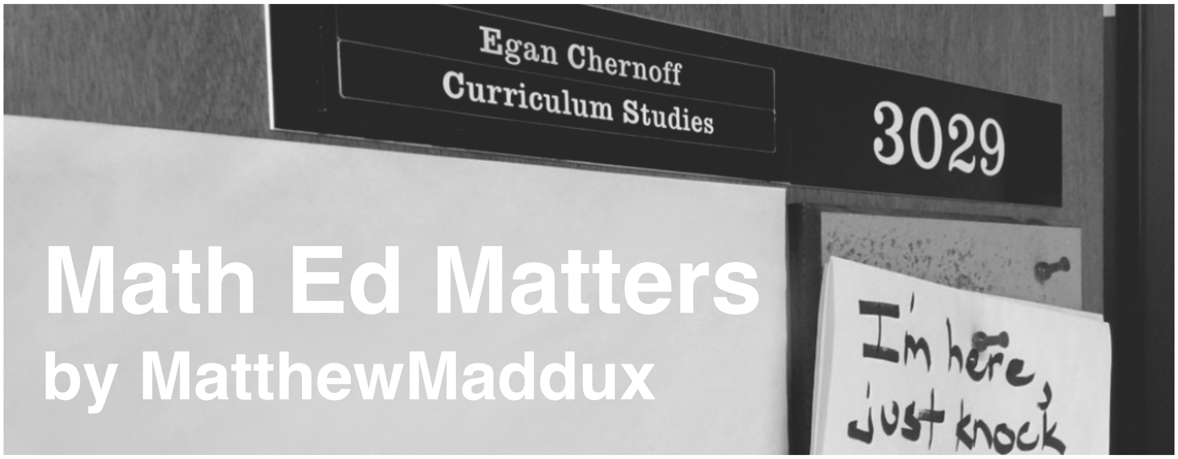
See http://www.cemc.uwaterloo.ca/contests/gauss.html

Sun Life Financial Canadian Open Mathematics Challenge
November 2017
Presented by the Canadian Mathematical Society

A national mathematics competition open to any student with an interest in and grasp of high school math. The purpose of the COMC is to encourage students to explore, discover, and learn more about mathematics and problem solving. The competition serves to provide teachers with a unique student enrichment activity during the fall term. Available in English and French. Written in November.

Approximately the top 50 students from the COMC will be invited to write the Canadian Mathematical Olympiad (CMO). Students who excel in the CMO will have the opportunity to be selected as part of Math Team Canada — a small team of students who travel to compete in the International Mathematical Olympiad (IMO).

See https://cms.math.ca/COMC



Math Ed Matters by MatthewMaddux *is a bimonthly column telling slightly bent, untold, true stories of mathematics teaching and learning.*

Subtraction: How the Hunted Became the Hunter

Egan J Chernoff

I have been listening to various versions of the phrase, "Damn kids these days, can't even make change without a calculator," for what has felt like my entire life. Kids, in this phrase, is a synecdoche for young people that work at, with, or behind a cash register or till. By the way, whether the register/till is at a grocery store, convenience store, drug store, movie theatre, restaurant, coffee shop, pub, bar, retail outlet, gas station, or wherever, it doesn't matter. What matters is that these young people and their inability to make simple change behind the till is seen as a measuring stick for the rapid decline of the teaching and learning of mathematics in our schools and our society as a whole. There's a lot to unpack here.

I should point out that I am not immune from the above conversation. In 2000, the nice lady conducting my interview for a bartending position at The Birchwood Motor Hotel (an hour north of Winnipeg on Manitoba Highway 59) pointed out that she was impressed that I had a BSc in mathematics. So impressed, actually, that she said that I would have no problem with the most important question of the interview. She asked, "If a bottle of beer costs $2.95 and a customer gave you a $5 bill, how much change do they get?" I promptly replied, "$3.05." Oops. Shocking, I know… a bottle of beer used to cost $2.95! Oh, I see, you're laughing at my incorrect answer. Stop laughing. Quizzically, the interviewer looked up from her sheet, but before she could look up the whole way I was able to tweak my answer to the correct amount of $2.05. We had a good laugh about me giving away the hard-earned money of the hotel owner, which was followed up by a series of questions involving the price of 2, 3, 5, and 7 beers, and the change that would be given back for various denominations of bills. Good news: I nailed this remaining portion of the interview and got the job.

I should point out that at the time, I wasn't the prototypical whippersnapper that most picture behind the till when they start to wax poetically about young peoples' inability to make change, and how this is a sign that the world is quickly going to hell in a hand basket. Quite the opposite. At the time of the interview, I was in my early-to-mid-twenties (which, yes, I did consider old at the time), had majored in mathematics at university, was very confident in my arithmetic (and mathematical) abilities, and, perhaps most importantly, was not fazed by and was able to catch little errors that arose during my mental math moments. Barring certain notable servers, one in particular, my mental math behind the bar would soon become the stuff of legend at the Birchwood Motor Hotel. It wasn't just the speed with which I could calculate the cost for big orders; rather, it was the confidence that I had in my answers that impressed my colleagues. That's enough about me.

I contend that when we talk about peoples' inability to make change, we have been too restrictive of the individuals under scrutiny. Ageist, if you will. Given the way our society is structured, it's true that there are predominantly young people behind the tills. The reason for this, for the most part, is simple: young people can get paid less. (This notion was expressed eloquently by podcaster Adam Carrola, who said that we live in a "minimum wage gilded cage.") The change-making ability of older individuals that work with/behind a cash register *must* be better. And, keeping in line with the current mathematics education zeitgeist in Saskatchewan and across Canada, there is a dominant working theory about why this must be so: given that the change-making individuals are older, they will have been taught mathematics *properly*—that is, in a traditional fashion. All of those lectures, worksheets, Mad Minutes, homework questions, quizzes, and tests concerning elementary arithmetic (adding, subtracting, multiplying and dividing) mean that elder folks should have no issues when tasked with elementary arithmetic tasks… The thing is, I have found quite the opposite to be true.

"Albeit unofficially, I have been testing the numeracy skills, specifically subtraction, of liquor store employees ever since I turned 19 years old."

Albeit unofficially, I have been testing the numeracy skills, specifically subtraction, of liquor store employees ever since I turned 19 years old (the legal age to purchase alcohol in British Columbia). Well, not exactly ever since. The years during which I conducted my research were 1996, 2000-2006, and 2010-2016. The years during which I did not conduct my research were 1997-1999 and 2007-2009. If you have spotted the pattern, then you know that I am not conducting my experiment this year, 2017, and for the next few years, 2018 and 2019. If I'm being honest, I'm not sure I am even going to pick the experiment back up in 2020-2026, because I'm not sure that I'm going to like the results. As I have alluded to, my experiment is a test in subtraction, but I would be remiss not to mention that it is also an exercise in pushing back against authority.

For a very long time, I was nervous about handing over my identification to a liquor store employee so that they could determine whether or not I could purchase alcohol (mostly beer). This stems, in part, from my efforts to buy beer (and coolers) before I was legally allowed to make these purchases. My nervousness manifested itself physically with an elevated heart rate, and sometimes, my palms would start to get a bit clammy. Mostly, though, my nerves would be exposed through the awkward back-and-forth exchanges with the employee that stood between me and the purchase of my alcohol. "Did you find what you were looking for?" the employee would ask as I slid my purchase across the counter, to which I would sheepishly reply, "Fine, thanks." My odd replies to questions that were not asked of me were a tell. They told the employees that I was nervous, which meant that I might be trying to purchase alcohol as a minor. "Can I see your ID?" the clerk would ask next. For whatever reason, perhaps due to a deep-seated fear of authority, my nervous replies did not disappear once after I was old enough to legally purchase beer throughout Canada and North America.

I should come clean. I said, earlier, that I conducted my subtraction research in 1996 and from 2000-2006, but that is not entirely true. It would be more accurate to say that I laid the ground work for my subtraction research during those years. What I noticed during that time was that the employees who were attempting to determine whether or not I was of age had a tough time with subtraction. Being born in 1977, the difficulties clerks had with borrowing in 1996 and from 2000-2006 was something that I noticed. I could see their mental arithmetic efforts in their faces. However, the context, the nerves, and the long line up of older people standing behind me meant that I would usually only make note of the subtraction troubles of the clerks after our exchange had completed. And, not making much money at the time, I focused my in-the-moment efforts on making sure that the clerk gave the right amount of change back. (Remember using cash to make purchases?!) Then, one day, one exchange changed everything for me.

I'm not sure what made me do what I did on that fateful day. It must have been a cocktail of getting older, being sick and tired of getting IDed, and being more confident with myself in life, in general. In 2006, a 29-year-old Egan J Chernoff was asked, "Can I see your ID?" as he was purchasing some beer. My response, as I was reaching into my wallet for my driver's license, was, "Yes, you may, but you're going to have to tell me how old I am…"

> "At that moment, it was abundantly clear to me that I had irreparably upset the balance of power between IDer and IDee."

At that very moment, it was abundantly clear to me, and the numerous people in the lineup behind me, that I had irreparably upset the balance of power between IDer and IDee. The hunted had become the hunter. The clerk, an elder gentleman, was a bit stunned with the question. I don't think he knew what to say at that point. He took my ID, looked at it, handed it back and said, "You're old enough." Not satisfied with the answer to my question, and channelling the nerves of all those people throughout the years who felt, and still feel nervous about handing over their ID even though they are of legal age, I doubled down: "How can you tell me I'm old enough without telling me how old I am?" His response, "You're old enough," was not satisfying. In that moment, I realized that I may have lost this battle. But it did not matter, because I was about to engage in a war with liquor store employees every time I got asked for my identification.

"Sure. Please tell me how old I am," is now my go-to response when I am asked for identification. The responses that I have been given over the years have been fascinating. Clerks are stunned when I ask this question. So are the other people in line. The most memorable response is also one that still bothers me to this day. At the very moment that I finished asking a nice lady behind the till to tell me how old I am—that is, to do a subtraction problem in front of me in real time—she looked like she had seen a ghost. My question must have taken her back to some dark place in her mathematical past. Perhaps she used to get singled out by her teacher to walk up to the board and do a subtraction problem in front of the rest of the class. Perhaps she had a tough time with her subtraction Mad Minutes. Who knows. I do know, though, that she became so pale that I told her I was just joking. But it didn't matter—the damage was done.

There are also employees who will indulge me in my request for subtraction. They are wrong much more often than they are right. The most common mistake is that they age me a decade older than I actually am. I contend that this mistake is because, as we all found out at some point, borrowing is difficult to do when engaging in mental arithmetic via the traditional algorithm learned in school. Checking the ID of someone born in 1977 in the year 2005 becomes, in the head of most clerks, 2005-1977. Utilizing the traditional algorithm involves crossing out the 5, then crossing out the 0, then crossing out the other 0, then crossing out the 2, putting a 1 where the 2 was, putting a 10 where the 0 was, crossing that out to put a 9 where the 10 was, then putting a 10 where the other zero was, then crossing that out to put a 9 where the 10 was, so that you change the 5 to 15 so that you can subtract 7. And that's the easy part. Clerks always get the last digit of my age correct. In this example, of course, that's 8. But, in this example, I would be incorrectly told that I was 38 years old, and not 28. This stems, I think, from the inability to keep track of the 0 turning to 10 and then turning to 9 in the tens column, which means subtracting 10-7 instead of 9-7. Whether young or old, those who employ the traditional algorithm for subtraction when unexpectedly tasked with mental arithmetic often, in my experience, get the answer wrong.

I'm still waiting for the day when I ask my question and the clerk standing between me and the purchase of my beer says something along the lines of, "Well, it's 2014 and you were born in 1977, but if it were 2017, then you would be 40, which means I need to account for three years, which makes you around 37, which means I don't have to worry about the month that you were born in for your stupid request, sir." Alas, no such exchange has taken place. Actually, I have found that my subtraction requests are being thwarted by a change that embraces the poor numeracy skills of North American citizens.

The days of having to complete a subtraction problem to determine whether or not someone is of legal age, for all intents and purposes, are over. The signs are everywhere. Actual signs.

I walked into a bar in Hawaii a few years ago and hanging on the wall was a sign, an electric sign that updated in real time, that read, "We I.D. Must be born on or before this date: [Month Day, Year]." Closer to home, similar signs are now hanging in the back rooms of pubs, bars, cold beer and wine stores, and liquor stores. Developments such as these, I contend, must be taken into consideration the next time you hear somebody utter the phrase, "Damn kids these days, can't even make change without a calculator."

Look, I agree: young people, those working behind tills and counters, should be able to make change. But I'm not sure I agree that their inability to make change is a true measuring stick for the rapid decline of the teaching and learning of mathematics in our schools and our society as a whole. The fact that I have consistently encountered relatively older employees who are unable to correctly answer a simple subtraction problem could easily be used to assert that we are, perhaps, looking through rose-coloured glasses when comparing the teaching and learning of mathematics of different generations. However, I would rather use this opportunity to shift the discussion from the change-making ability of young people to what's happening in the world around young people today.

> "Society is heading in a different direction—one less dependent on the elementary arithmetic skills of individuals."

Society is heading in a different direction—one less dependent on the elementary arithmetic skills of individuals. A society where restaurants calculate potential tips on your restaurant bill, even the 10% tip, for your convenience. A society where electronic signs on the grocery store shelves have already calculated the unit price for the items you want to purchase, for your convenience. A society where electronic signs post a birth date to let clerks and bartenders know if you are of legal age, subverting the need to do a simple subtraction problem. Our society is slowly eliminating the need for elementary arithmetic, one of the last bastions for teaching mathematics in school… whether those damn kids with those damn calculators can make change or not.

Egan J Chernoff (Twitter: @MatthewMaddux) is an Associate Professor of Mathematics Education in the College of Education at the University of Saskatchewan. Currently, Egan is the English/Mathematics editor of the Canadian Journal of Science, Mathematics and Technology Education*; an associate editor of the* Statistics Education Research Journal*; sits on the Board of Directors for* for the learning of mathematics*; serves on the International Advisory Board for* The Mathematics Enthusiast*; is an editorial board member for* Vector: Journal of the British Columbia Association of Mathematics Teachers*; and, is the former editor of* vinculum: Journal of the Saskatchewan Mathematics Teachers' Society.

Saskatchewan Mathematics Teachers' Society