The Variable

Presented by the
Saskatchewan Mathematics Teachers' Society

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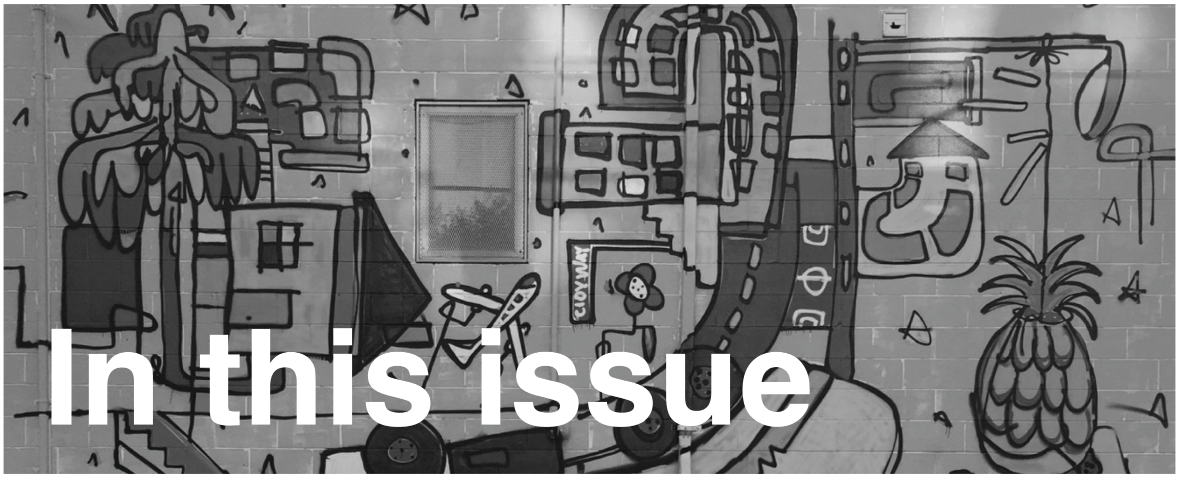


Cover Image

The mural featured in this month's cover photo was painted by Saskatoon-based artist Michael Remando and is located on Broadway Street in Saskatoon, Saskatchewan. See more of Michael's work at his website, www.rougegallery.ca/node/210, or follow him on Instagram @pineapplesforprimeminister.

SMTS Executive (2016-2018)

Notice to Readers

The views expressed in *The Variable* are those of the author(s), and not necessarily those of the Saskatchewan Mathematics Teachers' Society. In the publishing of any works herein, the editorial board has sought confirmation from all authors that personal quotes and examples have been included with the permission of those to whom they are attributable.

Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants, teacher educators, researchers, and students of all ages, although we encourage Saskatchewan teachers of mathematics as our main contributors. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Articles should be limited to 3000 words or less; authors should also include a photo and a short biographical statement of 75 words or less. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Call for Contributions

Did you just teach a great lesson? Or maybe it didn't go as planned, but you learned something new about the complexities of teaching and learning mathematics. Maybe you just read a book or attended a workshop that gave you great ideas for presenting a topic your students have always found difficult, or that changed your perspective about some aspect of teaching. **Why not share your ideas with other teachers in the province—and beyond?**

The Variable is looking for contributions from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants and coordinators, teacher educators, and researchers. Consider sharing a favorite lesson, a reflection, an essay, a book review, or any other work of interest to mathematics teachers in Saskatchewan. If accepted for publication, your piece will be shared with a wide audience of mathematics educators in Saskatchewan and beyond.

We are also looking for student contributions in the form of artwork, stories, poems, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom and their school, and for teachers to recognize their students' efforts during their journey of learning mathematics.

All work is published under a Creative Commons license. If you are interested in contributing or have any questions, please contact us at thevariable@smts.ca.

We look forward to hearing from you!


Saskatchewan Mathematics Teachers' Society
Spdu Professional Growth Partnerships
SELU
SASKATCHEWAN EDUCATIONAL LEADERSHIP UNIT
Leading Together
The SMTS, SELU and SPDU presents ...
#SUM2017
Who: K-12 teachers, coaches, consultants, coordinators, superintendents and directors
Where: TCU Place, Saskatoon
When: October 23-24, 2017
Cost: $315 (early registration) | $375 (regular)
Keynote Presenters
Lisa Lunney Borden, St. Francis Xavier University
Steve Leinwand, American Institutes for Research
Featured Presenters
Kristin Gray, Cape Henlopen School District, Illustrative Mathematics
Glenn Waddell, The University of Reno, Desmos Fellow
Egan Chernoff, University of Saskatchewan
Gale Russell, University of Regina
NCTM
AFFILIATE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Saskatchewn Understands Mathematics Conference 2017

Keynotes

Steve Leinwand

Practical Suggestions for Building a Powerful and Professional 2017-2018 To-do List

This fast-paced and example-laden pep-talk will discuss and model a set of instructional shifts that NCTM's Mathematical Teaching Practices and the quest for more effective instruction require us to consider in order to enhance our teaching and our students' learning.

Featured Session 1: Proving the Leadership Necessary for Making Mathematics Work for All Students (Part 1)
We know that effective programs of K-12 mathematics require informed and effective leadership. This part one of a two-part series of workshops will focus on specific understandings that every mathematics leader needs to have about effective mathematics programs, with a focus on high quality instruction, to be in a position to advocate for and support such programs.

Featured Session 2: Proving the Leadership Necessary for Making Mathematics Work for All Students (Part 2*)
We know that effective programs of K-12 mathematics require informed and effective leadership. This part two of a two-part series of workshops will focus on specific strategies and initiatives that every mathematics leader needs to establish, nurture and monitor to ensure that the effective mathematics programs discussed during part one are available to all students in every school. We'll take particular look at a range of collaborative structures that reduce professional isolation and support professional growth.

*attendance at Part 1 is not required to participate in Part 2

Lisa Lunney Borden

The Role of Mathematics Education in Reconciliation

The 2015 TRC final report that includes calls to action in response to the horrors of residential schools for Aboriginal Canadians that are focused on establishing a renewed relationships between non-Aboriginal and Aboriginal Canadians to "restore what must be restored, repair what must be repaired, and return what must be returned" (2015, p. 6). The TRC names the education system as having an essential role in repairing the damages caused by residential schools. Lisa will reflect on her 22-year career as a teacher and researcher working in Indigenous communities, primarily Mi'kmaw communities, to explore the role of mathematics education in reconciliation. She will share stories of hope and healing that have emerged through the Show Me Your Math program, inquiry projects, outreach programs, and teacher professional learning that give insights into how mathematics can aid in reconciliation.

Featured Session 1: Our Ways of Knowing: Teaching Math with Verbs and Space
Lisa will share a model for considering ways in which Indigenous languages, community values, ways of knowing, and cultural connections can impact mathematics learning for Indigenous learners. Participants will go more deeply into the pedagogical implications of this model that are linked to the ways of knowing that emerge from an understanding of the structure of Indigenous languages. We will engage in tasks that highlight the value of verbifying and spatializing mathematics teaching and learning. Examples will be drawn from Kindergarten to Grade 12 to highlight how these approaches span all levels.

Featured Session 2: My Elders were Mathematicians Too: The Value of Culturally-based Inquiry
Lisa will share the story of Show Me Your Math, a program that invites Indigenous students in Atlantic Canada to explore the mathematics that in inherent in community ways of knowing, being, and doing. She will share the history of this program, how it has changed over time to focus more on inquiry, and how it might be developed in other regions. We will explore examples of projects that have been completed, examine the benefits of these projects and discuss how such projects help to restore, reclaim, and return community knowledge that has been eroded by colonialism.

For more information on keynote and featured speakers, please visit:
http://smts.ca/sum-conference/sum-keynote-presenters

Message from the President

Welcome back! I hope that your first days back with students went smoothly, and that this year is full of exciting opportunities for you and your students.

Now that we're all back in the swing of things, it's SUM time! If you haven't heard about this year's very special edition of the Saskatchewan Understands Math (SUM) Conference, Leading Together, you a) must have not spoken to me this past year, and b) are going to be so excited when you find out all about it! It's no secret that I truly believe that SUM is one of the very best conferences for mathematics educators that you can attend in North America. At SUM, you can see the same presenters others pay thousands to see at large conferences (in crowds of 500 or more) right here at home during four (*four!*) hours of workshop time in a small-group setting, within a community of math educators who understand the unique demands of your classroom and curriculum.

Nothing about this is unusual, however, when it comes to SUM—this has been the norm year after year! What's so special, then, about SUM 2017? Some great partnerships, for starters. This year, we've partnered with the Saskatchewan Educational Leadership Unit (SELU) and the Saskatchewan Professional Development Unit (SPDU) to expand the scope of SUM Conference to offer sessions for system leadership of all levels, in addition to our regular offerings for K-12 classroom teachers. We are also very excited to be marking our first year with support from the Ministry of Education.

So, in addition to the regular reasons to attend SUM (see the previous issue of *The Variable*), there are a few more reasons to attend in 2017. At this year's conference, you can also expect:

- A focus on supporting leaders of all levels to prepare and/or support their systems in their work as they strive to meet the mathematics goals of the Saskatchewan Education Sector Strategic Plan (ESSP);
- An emphasis on increasing understanding of assessment and on responsive instruction to improve math outcomes for *all* students;
- A space for teachers and leaders to find out more about the "math at grade level" goal in the ESSP;
- In-depth discussion about how we can all actively answer the calls to action of the Truth and Reconciliation Commission, both by creating space for multiple ways of knowing in our classrooms and by using math as a way of understanding our shared history and present contexts.

I'm very good at lists, and I could definitely keep expanding this one. Instead, I'll leave you with one last great piece of news. As a response to the uncertainty and stressful financial times that school divisions found themselves facing last spring, **we have extended our early-bird pricing for SUM Conference until September 29**. So head over to our website (www.smts.ca/sum-conference/), check out the session descriptions, and register! If, for whatever reason, you will be unable to join us, we certainly hope to see you online, tweeting and learning with us using the #SUM2017 hashtag.

Have a wonderful September, everyone. See you at Leading Together: SUM 2017!

Michelle Naidu

Problems to Ponder

Welcome to the September/October edition of Problems to Ponder! This collection of problems has been curated by Michael Pruner, president of the British Columbia Association of Mathematics Teachers (BCAMT). The tasks are released on a weekly basis through the BCAMT listserv, and are also shared via Twitter (@BCAMT) and on the BCAMT website.

Have an interesting solution? Send it to thevariable@smts.ca for publication in a future issue of The Variable.

British Columbia Association of Mathematics Teachers

I am calling these problems ‘competency tasks’ because they seem to fit quite nicely with the curricular competencies in the British Columbia revised curriculum. They are non-content based, so that all students should be able to get started and investigate by drawing pictures, making guesses, or asking questions.

When possible, extensions will be provided so that you can keep your students in flow during the activity. Although they may not fit under a specific topic for your course, the richness of the mathematics comes out when students explain their thinking or show creativity in their solution strategies.

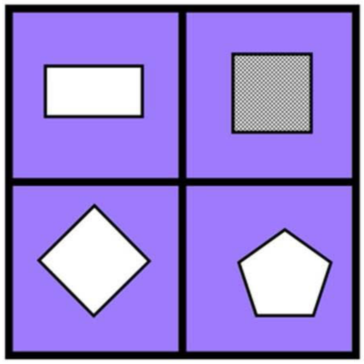
I think it would be fun and more valuable for everyone if we shared our experiences with the tasks. Take pictures of students’ work and share how the tasks worked with your class through the BCAMT listserv so that others may learn from your experiences.

I hope you and your class have fun with these tasks.

Michael Pruner

Primary Tasks (Kindergarten-Intermediate)

Which One Doesn't Belong?[1]
Display the image and ask the class: "Which one doesn't belong?"



This task encourages students to use descriptive language in their reasoning and to consider multiple possible answers. See wodb.ca for more details and more images.

Play With 60[2]
You have 60 items in a bowl. How could you arrange them to make it easier for a friend to count? Can you organize them in different ways?

You want to put the 60 items into bowls so that there is the same number in each bowl; how many different bowls will you need so that there are no items left over?

The Tortoise and the Hare[3]
The Tortoise has challenged the Hare to a hopping competition. The challenge is for the Hare to complete 3 equal hops and not land on a red square. Can the Hare succeed in this challenge?

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Can the Hare succeed in this challenge?

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

[1] Bourassa, M. (n.d.). Shape 2 [Digital image]. Retrieved from http://www.wodb.ca/shapes.html
[2] Problem sets. (2017). *Vector*, *58*(1), 47. Retrieved from http://www.bcamt.ca/wp-content/uploads/2017/05/581-Spring-2017.pdf
[3] *Adapted from* Tortoise and hare—revenge race. (n.d.). Retrieved from http://mathpickle.com/project/tortoise-and-hare-the-revenge-race-skip-counting-pattern/

Does the Hare ever fail? Try each of these:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Intermediate and Secondary Tasks (Intermediate-Grade 12)

The Shoe Sale[4]
You decide to take advantage of a buy *2 pair get 1 pair of equal or lesser value for free* sale at the local shoe store. The problem is that you only want to get two pairs of shoes. So, you bring your best friend with you to the store. After much deliberation, you settle on two pairs of shoes – a sporty red pair for $20 and a dressy black pair for $55. Your friend finds a practical cross trainer for $35. When you proceed to the check out desk the cashier tells you that your bill is $90 plus tax (the $20 pair are for free). How much should each of you pay? Justify your decision.

Elections[5]
There are two parties in an election: Red and Blue. There are only five people voting, and they are numbered 1, 2, 3, 4 and 5. What's interesting about this election is that the person's number counts as the number of votes that they cast. When the votes were counted:

- more people had voted for Red;
- Red had the more votes than Blue;
- if any one person had changed their vote, then Blue would have won.

What are all the possible ballot counts for this situation?

Extensions: What about 6 people, 7 people, and so on? What if there were three parties?

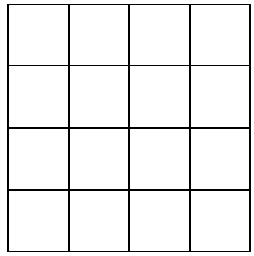
Truth-Tellers[6]
Truth-tellers always tell the truth and are marked with an 'X' on the grid below (*see next page*). Liars always lie and are marked with an 'O.' When asked the question, "Are you next

[4] Liljedahl, P. (n.d.). Numeracy tasks. Retrieved from http://www.peterliljedahl.com/teachers/numeracy-tasks

[5] Virtuous democracy. (n.d.). Retrieved from http://www.playwithyourmath.com/

[6] *Adapted from* Pinocchio's playmates. (n.d.). Retrieved from http://mathpickle.com/project/pinocchios-playmates/

to exactly two like yourself?" Everyone responded, "YES!" Where are the truth-tellers and the liars?



Extensions: Is this a unique solution? How do you know? What about a 5x5 grid, 6x6, …?

Michael Pruner is the current president of the British Columbia Association of Mathematics Teachers and a full-time mathematics teacher at Windsor Secondary School in North Vancouver. He teaches using the Thinking Classroom model where students work collaboratively on tasks to develop both their mathematical competencies and their understanding of the course content.

Mining Math Contests for Good Problems

Shawn Godin

Problem solving lies at the very heart of mathematics. As teachers, we want to help our students develop their higher-order thinking skills, which will be needed when they encounter real-world problems. As teachers, we also have a list, which can be quite lengthy, of topics that we must introduce in our courses. And, as *mathematics* teachers, we also have the added pressure that much of the content in our courses will be needed by students as they move on to the next level.

> "As teachers, we want to help our students develop their higher-order thinking skills. However, we also have a list, which can be quite lengthy, of topics that we must introduce in our courses."

Most problems in mathematics textbooks are fairly straightforward applications of the particular ideas presented in the section in question. Occasionally, there are a few "part C" problems, but in most cases, we have to search for things to stimulate our students. Fortunately, there is a vast collection of problems at your disposal in the form of problems from mathematics competitions.

There are a number of regional, national, and international mathematics competitions held each year, and many of them offer past contests for free on their websites. From a national point of view, there are the Canadian Mathematics Competitions organized by the Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo. The CEMC runs many mathematics and computer science competitions each year for students from Grades 7 through 12. In some contests, all of the problems are multiple choice, while others require full written solutions with marks given for clarity as opposed to only awarding full marks for the correct answer. Each year, well over 100 new problems from these contests are made available, adding to the almost 20 years' worth of problems already available on the CEMC website. These contests and other resources are available at www.cemc.uwaterloo.ca.

> "Fortunately, there is a vast collection of problems at your disposal in the form of problems from mathematics competitions."

The Canadian Mathematical Society (CMS) also supports mathematical enrichment in a number of ways. The CMS sponsors local, regional, and national math camps and activities, produces mathematical enrichment booklets (ATOM—A Taste Of Mathematics), and organizes several math contests. The Canadian Open Mathematics Challenge is open to all high school and elementary students. The top-performing students who write this contest are selected to write the Canadian Mathematical Olympiad (CMO), and in turn, the top-performing students of the CMO will represent Canada at the International Mathematical Olympiad (IMO) each year. Past contests, as well as links to other Canadian mathematics competitions and other resources can be found on the CMS website at cms.math.ca.

Another great source for mathematics competitions and other resources is the Art of Problem Solving website, artofproblemsolving.com. This site was started by a couple former IMO participants from the United States. It offers courses and books for problem solvers, as well as forums, places for sharing math contests from around the world, and problems from American mathematics contests going back to 1950.

Now that you have access to all these wonderful problems, what do you do with them? As the contest writers generally want students to experience some success, most include some problems that are quite easy. On the other hand, math contests are meant to be challenging, so some of the problems can be extremely difficult for students (and their teachers!). Looking at the problems on the easier side, you will find lots of questions that can be used as warm-ups with your classes. The warm-ups can serve the purpose of just doing a problem that should be routine, such as:

> *If $x = 3$, $y = 2x$ and $z = 3y$, then calculate the value of z.*
>
> (2016 Fermat Contest (CEMC), question #1)

We could also use the warm-ups to reinforce an important idea that may not be an explicit curriculum outcome, such as:

> *A triangle initially has area 100 square units. If its base is increased by 10% and its altitude is decreased by 10% what is the new area?*
>
> (2016 Manitoba Mathematical Contest, question #2 (a))

Problems like this one are good for reinforcing the idea that increasing or decreasing a value by a certain percentage is a *multiplicative* action, not an additive one. As a result, an increase followed by a decrease of the same percentage, or vice versa, does not bring you back to where you started. This problem will probably elicit various types of solutions from students. Students who are comfortable with abstraction will work with variables and come to a solution quickly. Others may find the dimensions of a triangle with area 100 square units and work with them to do their calculation.

The problem below is good for differentiating between linear and non-linear relations.

> *A 2 cm cube (2 cm × 2 cm × 2 cm) of silver is worth $40. How many dollars is a 3 cm cube (3 cm × 3 cm × 3 cm) of silver worth?*
>
> (34th New Brunswick Mathematics Competition, question #9)

Many students will quickly, and erroneously, deduce that a 1 cm cube would be worth $20 so a 3 cm cube will be worth $60. If students are struggling, offering manipulatives, such as linking cubes, may help. It would be interesting to see if students who determined the correct solution, if pressed for an alternate solution, would come up with a similarity argument, or a solution based on percentage increase of the dimensions, like in the previous problem.

A problem can also be used to reinforce the importance of reading information and to give students an opportunity to deal with known tools in an unfamiliar situation. Take the two problems below

> *If the symbol* $\begin{matrix} p & q \\ r & s \end{matrix}$ *is defined by* $p \times s - q \times r$*, then what is the value of* $\begin{matrix} 4 & 5 \\ 2 & 3 \end{matrix}$*?*
>
> (2016 Pascal Contest (CEMC), #6)

Roberta Beaver has purchased an old computer that only allows one digit after the decimal point in any calculation. Anything after that digit is removed. Sometimes this results in an error which is the difference between the stored value and the exact value. For example, if we try to compute $\frac{7}{5}$ *on Roberta's machine, this will be stored as 1.4 which is the exact value of* $\frac{7}{5}$*. This gives an error of 0. However, if we compute* $\frac{7}{4}$*, this will be stored as 1.7 since* $\frac{7}{4} = 1.75$ *and "5" will be removed from the end. This gives an error of 0.05. Extra digits are removed after every operation. For example, when Roberta computes* $\left(\frac{3}{2}\right) \times \left(\frac{2}{3}\right)$ *she computes* $\frac{3}{2}$ *to give 1.5, then* $\frac{2}{3}$ *to give 0.6, and then* 1.5×0.6 *to give 0.9. This gives an error of 0.1. If Roberta calculates* $\left(\left(\frac{10}{3}\right) \times \left(\frac{10}{3}\right)\right) \times 9$*, what is the error?*

(2016 Beaver Computing Challenge (CEMC), question A2)

In both of these questions, the students are just doing addition, subtraction, multiplication, or division. Students have to carefully follow the instructions, either in algebraic or verbal form, to come up with their answers. These are both great exercises in decoding.

We can also use contest problems to meet curriculum goals. For example, if we want to introduce, or remind the students of the Pythagorean theorem, then the following problem engages their prior knowledge and gets them using a tool that they will soon need:

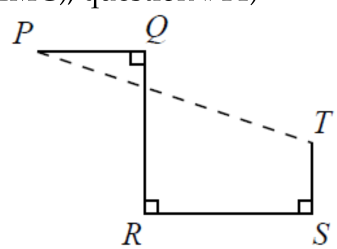
What is the expression $\sqrt{5^2 - 4^2}$ *equal to?*

(2016 Cayley Contest (CEMC), question #3)

Then students have to put in a bit of thinking to attack the problem below, which also involves the Pythagorean theorem:

In the diagram, PQ is perpendicular to QR, QR is perpendicular to RS, and RS is perpendicular to ST. If $PQ = 4, QR = 8, RS = 8$ *and* $ST = 3$*, then what is the distance from P to T?*

(2016 Pascal Contest (CEMC), question #14)



Some contest problems mix concepts in a way that is probably not familiar to most students. The task, then, becomes recognizing how the problem is related to things the student already knows. For example, if you are looking at systems of linear equations with your students, the following problem is a nice, straightforward word problem:

Liza has a row of buckets. The first bucket contains 17 green discs and 7 red discs. Each bucket after the first contains 1 more green disc and 3 more red discs than the previous bucket.

(a) Which bucket contains 16 red discs?
(b) In which bucket is the number of red discs equal to the number of green discs?
(c) There is a bucket in which the number of red discs is twice the number of green discs. In total, how many discs are in this bucket?

(2016 Galois Contest (CEMC), question #1)

On the other hand, the next two problems have our system written in an unfamiliar form.

Determine the pair of real numbers x, y which satisfy the system of equations

$$\begin{aligned} \frac{1}{x}+\frac{1}{y} &= 1 \\ \frac{2}{x}+\frac{3}{y} &= 4 \end{aligned}$$

(2016 Canadian Open Mathematics Challenge (CMS), question A3)

Solve the system

$$\begin{aligned} 3^{2x-y} &= 27 \\ 2^{3x+2y} &= 32 \end{aligned}$$

(The 33rd W. J. Blundon Mathematics Contest, question #1)

The first problem can be solved in a number of ways. We can multiply both equations by xy and then solve for either x or y in terms of xy which, since neither of x or y can be zero, allows us to solve for the other variable. We could also use a change of variable by letting $X = \frac{1}{x}$ and $Y = \frac{1}{y}$ and solving for X and Y, then getting x and y by finding the reciprocal of these. This is a useful technique that can frequently be used to help the solver see some underlying structure that may not be obvious at first sight. We could also forego the change of variable and solve directly for $\frac{1}{x}$ and $\frac{1}{y}$. The change of variable allows us to solve the problem when the direct method isn't obvious.

In the second problem, students need to recognize that the constant terms need to be written as powers, after which the problem can be rewritten in a more familiar form.

Another thing that we find in contest problems is situations where concrete examples of ideas may be harder to deal with than abstract ones. For example

Evaluate the following product:

$$\frac{4}{3}\cdot\frac{9}{8}\cdot\frac{16}{15}\cdot\frac{25}{24}\cdot\ldots\cdot\frac{2015^2}{2015^2-1}$$

(2014-2015 Nova Scotia Math League, Game 3, question #8)

Students may be reluctant to factor $2015^2-1=(2015-1)(2015+1)=2014\times2016$ since there are no variables involved, but it is good for them to recognize that it is exactly the same idea as when variables are involved.

Another thing that we can do with contest problems is adjust them for our own means. This may mean taking a problem and changing the level of difficulty, or even the context. The following problem

> *Find the sum of all values x such that* $(x^2 - 7x + 11)^{x^2+3x-10} = 1$.
>
> (2016 British Columbia Secondary School Mathematic Contest, Senior Preliminary Round, question #8)

requires students to think about properties for powers while also considering properties of polynomials. If our class doesn't have the algebraic skills to tackle this problem, we can lower the level a bit, keeping the idea of the properties of powers intact, as well as some of the algebra. For example, if we change the equation to $(2x - 17)^{3x-12} = 1$, students still have to consider the case where the base is 1, the case where the exponent is 0 (and, hopefully, the base isn't 0) and the case where the base is -1 (and, hopefully, the exponent is an even integer). You could similarly make the problem more difficult by involving whatever functions you like in the base and exponent.

Similarly, the following problem

> *What can you say about four consecutive integers? Generate as many **mathematically significant** statements as possible. e.g., Saying that all four are numbers is **not** mathematically significant. e.g., Saying that their sum will always be even **and** that their sum will always be a multiple of 2 are not distinct statements.*
>
> (2016 Saskatchewan Math Challenge, Grade 9-10 Team Exam, Round 2)

is a nice open-ended exploration. We could use it as a template for our own ideas, such as, "What can you say about the linear relations $y = ax + b$ and $y = bx + a$?". The possibilities are endless.

The last two examples, as well as many of the other problems we have looked at, also serve as great springboards for students to ask their own questions and conduct their own investigations. George Polya, in his ground-breaking book on problem solving *How to Solve it*, broke the problem-solving process down into four steps:

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Reflect on the process

For me, reflection is essential. However, many students often skip reflection once they have found "the answer." The reflection step itself can be broken into many different sub-steps. Primarily, when we have finished solving a problem, we should check the reasonableness of our answer. Another important step is to think of other ways that the problem could be solved. Many problems will have more than one solution, and each solution tells a different story. Finally, we have the "mathematician" step where we ask questions such as: "How does this connect with other problems/knowledge?", "Why did we get this result?" and "What would happen if…?". This leads to extension, generalization, and forging our own path into mathematical exploration and discovery.

> "After we have solved a problem, we should think of other ways it could be solved. Many problems will have more than one solution, and each one tells a different story."

For the last 11 years, I have been writing a column, *What's the Problem?*, on mathematics problems and their solutions for the Ontario Association for Mathematics Education's journal, *The Gazette*. The editors of *The Variable* invited me to contribute to this periodical as well, and I agreed. I am looking forward to bringing my new column, *Alternate Angles*, to you, readers of *The Variable*. As mathematics is not a spectator sport, I will always assign some "homework" at the end of each column so that you can play with the problem that I will talk about in the next issue before you get to read about my own explorations. In that light, here is your first problem:

$$\textit{Evaluate } \left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)^2.$$

Play with the problem to see what you can do with it. Until next issue, happy problem solving!

Shawn Godin teaches and is a department head at Cairine Wilson Secondary School in Orleans, Ontario. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests.

Spotlight on the Profession

In conversation with Dr. Lisa Lunney Borden

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Dr. Lisa Lunney Borden, who we look forward to welcoming this fall as a SUM Conference 2017 keynote presenter.

Lisa Lunney Borden is an Associate Professor of mathematics education at St. Francis Xavier University in Canada with a particular focus on Equity in Mathematics. Having taught 7-12 mathematics in a Mi'kmaw community, she credits her students and the community for helping her to think differently about mathematics teaching and learning. She is committed to research that focuses on decolonizing mathematics education through culturally based practices and experiences that are rooted in Aboriginal languages and knowledge systems. Lisa is equally committed to mathematics outreach through programs such as Show Me Your Math *that was developed with David Wagner, Newell Johnson, and a team of teachers from Mi'kmaw Kina'matnewey schools. This program invites Indigenous youth to find the mathematical reasoning inherent in their own community context. Lisa is a sought after speaker on Indigenous mathematics education, working with mathematics educators across Canada as well as internationally.*

First things first: Thank you for taking the time for this interview!

Your research, coming on the heels of 10 years of teaching mathematics in a Mi'kmaw school in We'koqma'q, Cape Breton, Nova Scotia, focuses on culturally responsive mathematics curriculum and pedagogy. You have paid particular attention to the role that language plays in the teaching and learning of mathematics, and in particular, to ways in which shifting the language in our classrooms can support Aboriginal students in learning mathematics (e.g., Lunney Borden, 2011, 2013). For example, in Lunney Borden (2011), you describe the strategy of 'verbifying' mathematics—in other words, shifting your way of explaining concepts to be more consistent with the verb-based linguistic structures of Mi'kmaq—as a way of supporting Mi'kmaw students in mathematics learning.

How might teachers of Aboriginal students in other parts of the country (e.g., Saskatchewan), or more generally of students whose home language is not English, apply this work to their own local contexts?

Ways of knowing are embedded in language, and listening to how students talk about concepts in mathematics can help teachers think about how students are thinking about that concept. All Indigenous languages in Canada are verb-based, which means that focusing more on processes and motion can be of benefit to many Indigenous students. Leroy Littlebear talks about the sense of flux that is embedded in Indigenous languages and connected epistemologies, and this flux is more in line with a verb-based approach to learning mathematics. For this reason, verbifying school mathematics might feel more culturally consistent for Indigenous students; however, it may actually prove to be better for all students. Mathematical reasoning is about processes and change, and verbifying captures this essence more so than the noun-dominant approach we often see being used in school. I believe that most mathematicians would tell you that when they do mathematical work, they look at what is happening, what is changing, what happens when you change parameters, and so on—these explorations are all about processes.

> "Mathematical reasoning is about processes and change, and verbifying captures this essence more so than the noun-dominant approach we often see being used in school."

I don't think any mathematician set out to discover irrational numbers, quadratic functions, and so on; rather, they played with numbers and number patterns to see what they do under certain conditions, or how the patterns form; then, once they figure it out, they name it. They name it so they can do new things to it. This is very consistent with the process of verbification, and I think consistent with what real mathematicians do in their work.

In Lunney Borden and Wiseman (2016), you mention the success of Mi'kmaw Kina'matnewey (MK), a collective of Mi'kmaw communities in Nova Scotia, which has maintained a graduation rate between 87% and 89% in the past 5 years. This stands in stark contrast to the national graduation rates for Aboriginal children, which are reported to be around 48% (Assembly of First Nations, as cited in Lunney Borden & Wiseman, 2016). What has contributed to this success, and how has the Show Me Your Math program—which you developed in collaboration with teachers and elders in MK schools—played a role?

I don't think I can assign any credit to Show Me Your Math for the high graduation rates. Rather, this program emerged in a context that supports those high graduation rates. Simply put, both things happen because of a relentless commitment amongst Mi'kmaw people to decolonize education in their own communities. Mi'kmaw Kina'matnewey has worked very hard to build capacity for jurisdictional control of education in their communities. They have worked with universities, like StFX, to prepare teachers from the community to take on the teaching jobs in community schools. There have been over 135 Mi'kmaw BEd grads from StFX alone since 1996, and the majority of these teachers are working in MK schools. Many of these teachers have completed MEd degrees, certificate programs in Mi'kmaw language and mathematics, and other graduate programs. They have moved into leadership positions in schools and at the MK office. As such, you can find Mi'kmaw educators in all levels of MK education who speak the language, know the community contexts, understand the ways of knowing, being and doing of the community, and who are deeply committed to decolonizing education through inclusion of what Orr, Paul, and Paul (2002) called cultural practical knowledge. They bring stories of community into the classroom and allow these stories to speak back to the dominant narratives in the curriculum that have, for far too long, privileged Eurocentric thought.

When David Wagner and I invited MK teachers to a planning session to talk about Show Me Your Math in the fall of 2006, the room had teachers who spoke the language, who were deeply connected to community cultural practices, and who knew exactly where this idea could go. What keeps Show Me Your Math going is this commitment and a desire amongst these teachers to learn about ways of reasoning that are embedded in cultural practices and align with what we teach as mathematics in schools. This passion is also what helps students succeed in schools where they get to be themselves, learn in culturally consistent ways, and be proud of being Mi'kmaq. There are many interconnected factors that bring about these high graduation rates, but they all are rooted in the capacity development that has been such a central focus of MK.

Much of your work has been drawn from your experiences and work in small Mi'kmaw communities in Nova Scotia, and has emphasized the importance of respecting and supporting children's culture and language as a key factor in improving outcomes for Aboriginal students (e.g., Munroe, Lunney Borden, Murray Orr, Toney, & Meader, 2013). You have presented compelling examples of how mathematics can emerge from Indigenous contexts, rather than simply being imposed upon Indigenous artefacts (e.g., Lunney Borden & Wiseman, 2016; Munroe et al., 2013). However, this is clearly a particular challenge for teachers in urban settings, where a classroom typically includes students with a variety of cultural backgrounds and cultural experiences.

How might (mathematics) teachers in such settings honour students' culture and language without resorting to sweeping generalizations and/or tokenization of Aboriginal cultures, perspectives, and ways of knowing—that is, to the oversimplification of complex ideas that Edward Doolittle has dubbed the "cone on the range" approach (2006, p. 20)?

There are school districts in New Brunswick that are doing Show Me Your Math with all students, so *every* student is expected to learn about ways of reasoning that might be mathematical in their own cultural context; then, they share across cultural contexts. This has proven to be successful in promoting cross-cultural conversations that allow all students to see what we might call mathematical reasoning as a part of their cultural heritage. It also helps all students to see that mathematics is a human endeavour and that mathematical reasoning has emerged in many contexts. So that is one way to avoid trivialization—let all children explore their own interests and share their own stories.

> "Whether we are in an urban or rural setting, there is merit in learning about the Indigenous knowledges and technologies that existed long before settlers arrived on these lands."

That being said, I believe that whether we are in an urban or rural setting, there is merit in learning about the Indigenous knowledges and technologies that existed long before settlers arrived on these lands. Both Indigenous and non-Indigenous students would benefit greatly from learning about the Indigenous knowledges and how these ways of knowing, being, and doing allowed for settlers to survive in these lands. A big part of reconciliation is helping non-Indigenous Canadians learn about Indigenous peoples and knowledge systems, treaties that govern this land, and so on. When we center Indigenous knowledges and practices in our classroom as a starting point for learning mathematics, we are honouring the knowledge that springs from this place, and all children can benefit from this decolonized approach to learning.

For example, when I begin with a story about the late Dianne Toney who made quill boxes and knew that to make a ring for a circular top, she would measure with the wood strip three times across the circle and add a thumb width to make the ring, I can introduce an investigation for learning about pi that begins with Indigenous knowledge, rather than beginning by privileging a Eurocentric voice. All students see that this knowledge was passed down through generations of Mi'kmaw people and did not come from a Greek mathematician. Sure, the Greeks were interested in this relationship, too, but this lesson shows that they were not the only ones who knew this relationship. Eurocentric approaches have privileged only Eurocentric knowledge and have attempted to erase evidence of similar knowledge in non-European cultures.

> "When we take opportunities to raise up Indigenous knowledges and counter the discourse of European superiority, we are doing decolonizing work that benefits everyone and moves us closer to reconciliation."

The Truth and Reconciliation Commission has said that reconciliation is about respect, which requires all Canadians to challenge the notions of European superiority and Aboriginal Inferiority that shaped the colonization of these lands. When we take opportunities to raise up Indigenous knowledges and counter the discourse of European superiority that still permeates our system, we are doing decolonizing work that benefits everyone and moves us closer to reconciliation. How we do that without trivializing is by being committed to first unpacking our own privilege, examining our own place in the history of colonialism, and then being open to learning in honest and sincere ways. I think teachers need to build good relationships with Indigenous communities in their local area, whether urban or rural, and make a point to learn with their students.

Your work has frequently dealt with the issues of how mathematics curriculum and instruction can be more aligned with Aboriginal perspectives and ways of knowing. How might classroom assessment, too, be more aligned with Aboriginal perspectives on education? (This seems to be a particular challenge, given that Aboriginal worldviews emphasize holistic and interconnected knowledge, while Western assessments have often tested discrete skills in situations that are decontextualized from context.)

Well, I would again say that a move toward more holistic and applied knowledge would be a better way to assess every student. I don't think such a narrow focus on assessment is helpful for anyone. We want to educate a generation of children who can think and solve problems. We want them to be able to apply their learning in meaningful ways so that they are prepared to tackle the real problems of the world. Personally, I am much more interested in knowing whether or not a student can explain what multiplication means and show how it might be represented in multiple ways than I am in whether or not they can recite facts. Machines can do calculations, but humans need to tell them what calculation is needed in what context. That comes from meaningful experiences with various contexts and representations and not from memorization. Inviting a student to explain how a calculation might be used in a situational story will tell you more about their understanding of that calculation than giving a quick answer.

I'm not saying we don't want students to have fluency with numbers—of course we do, but fluency doesn't come from memorizing facts you do not understand. Consider, for example, a task where students are given 24 squares and are asked to make all the rectangular areas they can make using all 24 squares. Students will begin to see that 24 can be represented by 2 rows of 12 or 3 rows of 8 or 4 rows of 6 and so on. This is a holistic approach to working with quantity in a multiplicative way that allows students to see the factors of 24, and this is what will allow them in year to come to think flexibly about 24 in multiplicative contexts. This is how fluency develops. This is how we should be assessing.

In Lunney Borden and Wiseman (2016, p. 143), you write: "We have learned from Aboriginal colleagues that teaching and learning are fundamentally about relationships—an idea also deeply embedded in mathematics and science." And yet, Western mathematics and science have also been instrumental in concealing *the relationship between people and the natural world via knowledge and tools that allow humans to control certain natural phenomena and dissociate themselves from their environment.*

In light of this, could you elaborate on the connection between mathematics and Aboriginal philosophies?

As a non-Indigenous person, I don't think I can do justice to discussing Indigenous philosophies—I would invite folks to read work from Greg Cajete, Leroy Littlebear, Marie Battiste and others—but what I know for sure is that we would all benefit from being more connected to our world and to one another. It is easier to destroy the land when we feel no kinship connection to it. It is easier to commit atrocities against another nation when we do not see a kinship relationship with the people of that nation. Connecting with one another can certainly benefit our world and help us to think more deeply and responsibly about how we live with the land and ensure its survival for generations to come.

I think there are aspects of mathematics that can help children to see and value these connections. For example, mathematical modelling of complex problems is one way we can help students connect mathematics with the world around them. I work with colleagues on an outreach program called Connecting Math to Our Lives and Communities, where we engage Mi'kmaw and African Nova Scotian youth in activities that help them connect math to things going on in communities right now. For example, we have done work on water security in First Nations communities in Canada. One of our communities was under a boil-water order at the time, which has been an ongoing issue in that community. We used water security to learn about this issue and see how math can be used to model some of the issues related to water security in communities.

> "Being mindful of connections to our world will ensure that more students see the potential power of mathematics to support their own communities."

Much of mathematics emerged in contexts where humans were seeking to solve real problems in their communities. When we teach that math is not connected, we deny this human activity and make math seem mysterious and magical, and that doesn't help students to learn it and use it meaningfully. Being mindful of connections to our world will ensure that more students see the potential power of mathematics to support their own communities.

Lastly, our readers are likely aware that you will in Saskatoon this October to present as a keynote speaker at our very own Saskatchewan Understands Math (SUM) Conference. (We can't wait!) We don't want to spoil the surprise, but could you give our readers some insight into what you will be discussing during your sessions?

I will be doing a lot, based on the number of descriptions I sent in! ☺ My keynote will focus on the role of mathematics education in reconciliation, so I will share ideas about how we decolonize mathematics teaching and learning. I will also do a featured session where I will talk about a framework I developed in my doctoral work that draws upon Indigenous language to inform pedagogy—in particular, we will talk about verbifying and spatial reasoning. Two follow-up sessions will focus on what this looks like in early number work and in multiplication and division. I will also have a session about the Show Me Your Math program.

Thank you for taking the time for this conversation. We look forward to continuing the discussion at SUM 2017!

Ilona Vashchyshyn

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Making Homework Matter to Students [1]

Lee Walk and Marshall Lassak

During my teaching career (author Walk), I have been frustrated with student assignments being handed in incomplete, rushed through, or not at all. It made me wonder what students saw as the purpose of homework. I tried to impart the idea that the purpose of homework was to help students improve their understanding of mathematical concepts, practice skills, and act as a formative assessment that could help me see what they currently comprehended. Unfortunately, many students seemed to think that homework was just another unpleasant task to finish as quickly as possible without thinking deeply about what they were doing.

> "Many students seemed to think that homework was just another unpleasant task to finish as quickly as possible without thinking deeply about what they were doing."

Two studies (Trautwein, 2007; Dettmers et al., 2010) show a positive correlation between high-quality homework and mathematics achievement. Students who completed their homework assignments scored better on assessments. However, these studies also showed no relationship between time spent on homework and resulting student achievement. This helped verify that although homework can be a valuable tool for learning, more time spent on daily homework is not necessarily a good idea.

In continuing to research effective homework styles and implementation, I found in many studies that student collaboration was important in developing mathematical reasoning. Wieman and Arbaugh (2014) discuss how students could use homework as an opportunity to work together. However, they also emphasize that teachers must be clear about their goals and expectations for student homework:

> Students may think that they are supposed to be able to complete all the homework problems quickly and easily, and that referring to their notes or asking a friend for help is somehow "cheating." (p. 162)

Students must understand the difference between asking another student for help in thinking through a problem and simply asking for the answer. Homework does not need to be completed independently if the primary goal is to learn about mathematics.

Levels of Demand

Smith and Stein (1998) argue that the highest learning gains for students result from engagement in high levels of cognitive thinking and reasoning. They break down tasks in terms of four categories of cognitive demand:

1. Memorization
2. Procedures without connections to concepts or meaning
3. Procedures with connections to concepts
4. Doing mathematics

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Keeping in mind the appropriate level of homework challenge, it appears that most homework tasks should be in the third category (procedures with connections to concepts). The first two categories are considered to have a lower-level demand for students because they can be solved with limited or no cognitive demand. The third and fourth categories of tasks require deeper thinking and understanding. These tasks might be more complex and often have multiple solution paths.

Although the fourth level may be appropriate for classroom learning, it is likely to be too difficult for homework on a regular basis and could negatively impact student effort.

Changing My Approach

All this research led me to wonder if my students would be more successful in completing and learning from their homework if they were given fewer problems with a higher level of cognitive demand. I already knew that some students have difficulties with procedural questions and that even those students who are able to answer such questions often have a difficult time explaining the reasoning for their methods.

> "All this research led me to wonder if my students would be more successful in completing and learning from their homework if they were given fewer problems with a higher level of cognitive demand."

When teaching prealgebra to my eighth-grade students, I typically assign skill-and-drill questions accompanied by one short-response problem. Skill-and-drill problems give students repeated practice of a particular procedure and are intended to help them gain fluency. The majority of my students completed the homework problems, but those who often did not explained that they did not have "enough time" or did not understand what to do. These students were typically habitual offenders in failing to complete their assignments, and the homework never seemed important to them. I tried to motivate my students and access self-motivation by putting more control of the homework into their hands through additional time and choices.

Focusing on the algebraic concept of solving linear equations, I implemented a change in my homework style for several weeks. Manipulating and solving equations is a central concept for my students, and it is a skill that many have difficulty mastering. I thought this would be a good time to provide a better homework experience. My class received "cognitive" homework with fewer problems that had a higher level of demand. Weekly homework assignments contained suggestions about the number of questions to be completed each day. The presentation of the material for the unit and the structure of the classroom still allowed for time spent reviewing homework questions.

In changing the type of homework, I tried not to change my method of assessment or how I integrated homework into my lessons. I continued to collect and check homework for completion and accuracy. Students were allowed and encouraged to ask questions about homework ideas as part of daily lessons. Weekly quizzes were used to help determine student progress throughout the unit.

I emphasized that the new homework had fewer problems for them to complete and that daily assignments were only suggestions; nothing was due until the end of the week. This meant that if they were busy on a given night, they could get it done later in the week. I tried to motivate them by putting more control of the homework into their hands.

To foster student cooperation, I told students that they could ask me for help but reminded them that they could also rely on their classmates. On several occasions, I told students that they could work with a partner or in small groups to work out problems at the end of class. The class also engaged in think-pair-share activities to jumpstart thinking and discussion.

The new homework problems I created and assigned were influenced by suggestions obtained from reading work done by Wieman and Arbaugh (2014); Lange, Booth, and Newton (2014); and Friedlander and Arcavi (2012), all of whom offered advice on particular homework style questions as well as questions to elicit the type of thinking and activity advocated by Smith and Stein (1998).

Fig. 1 Students were given this format and asked to find and fix the mistake.

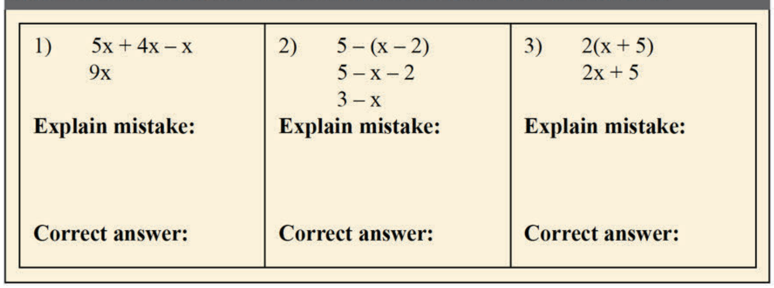


Fig. 2 Jennifer produced this explanation.

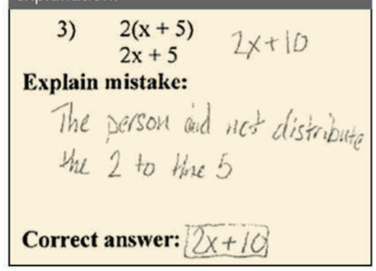


Fig. 3 Connor found these errors; he was unable to obtain the correct answer for problem 5.

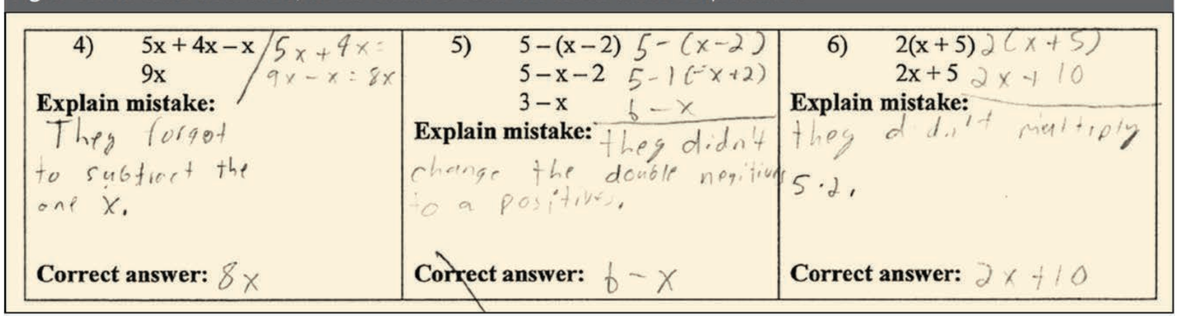
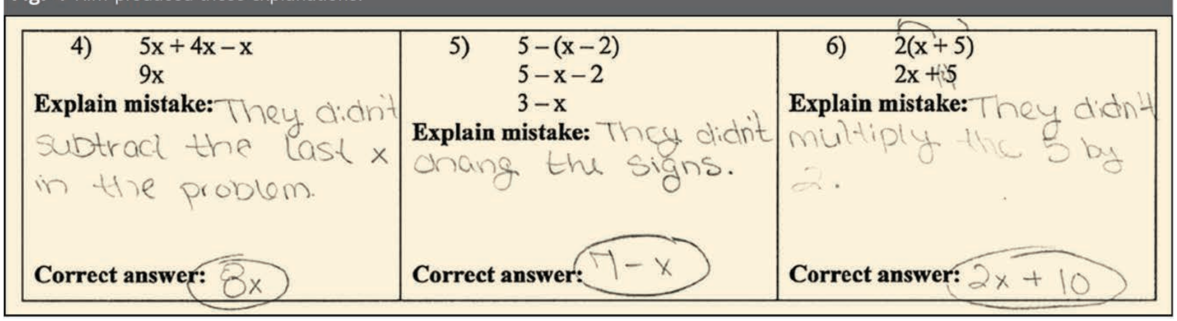


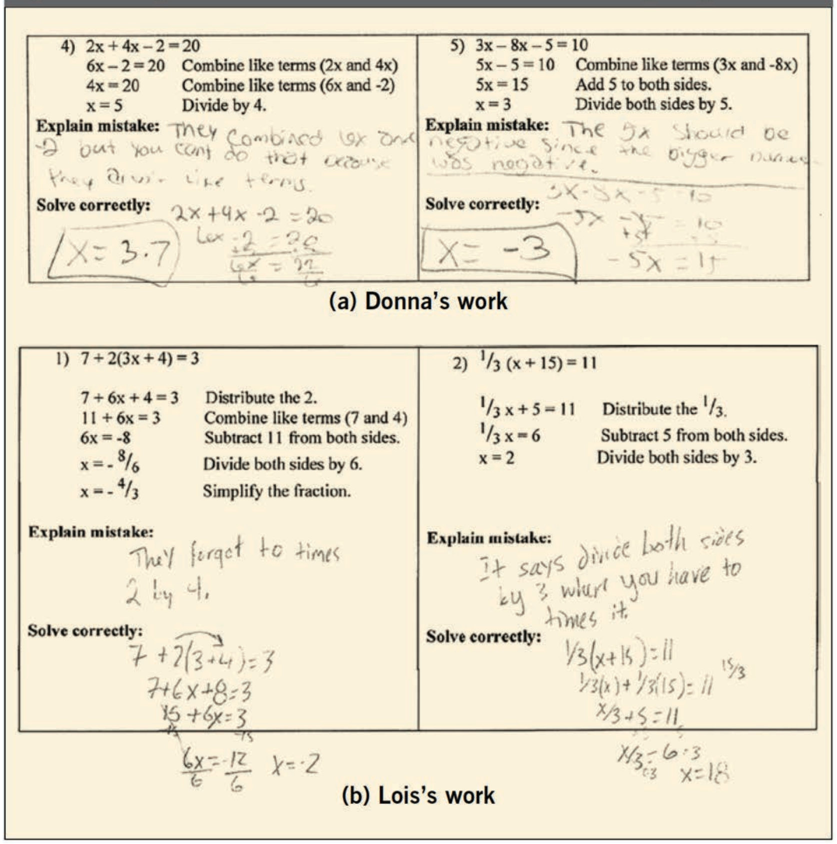
Fig. 4 Kim produced these explanations.



Find and fix the mistake. Find and Fix the Mistake problems (see Figure 1) were the most commonly used tasks throughout the new homework implementation. Students were instructed to identify the mistake, explain it, and simplify the expressions or solve the equations correctly. One advantage of these problems was that they required students to use correct mathematical vocabulary. Jennifer (see Figure 2) includes the term distribute in her response, which was discussed several times in class.

Connor (see Figure 3) and Kim (see Figure 4) both demonstrated their procedural knowledge; writing about the problem helped them participate in class discussion. During this time, I found that students were more confident in class discussions when they had had a chance to think before class about these ideas. Rather than just solving the equations with varying levels of procedural fluency, students were thinking more about the problems and how to explain errors.

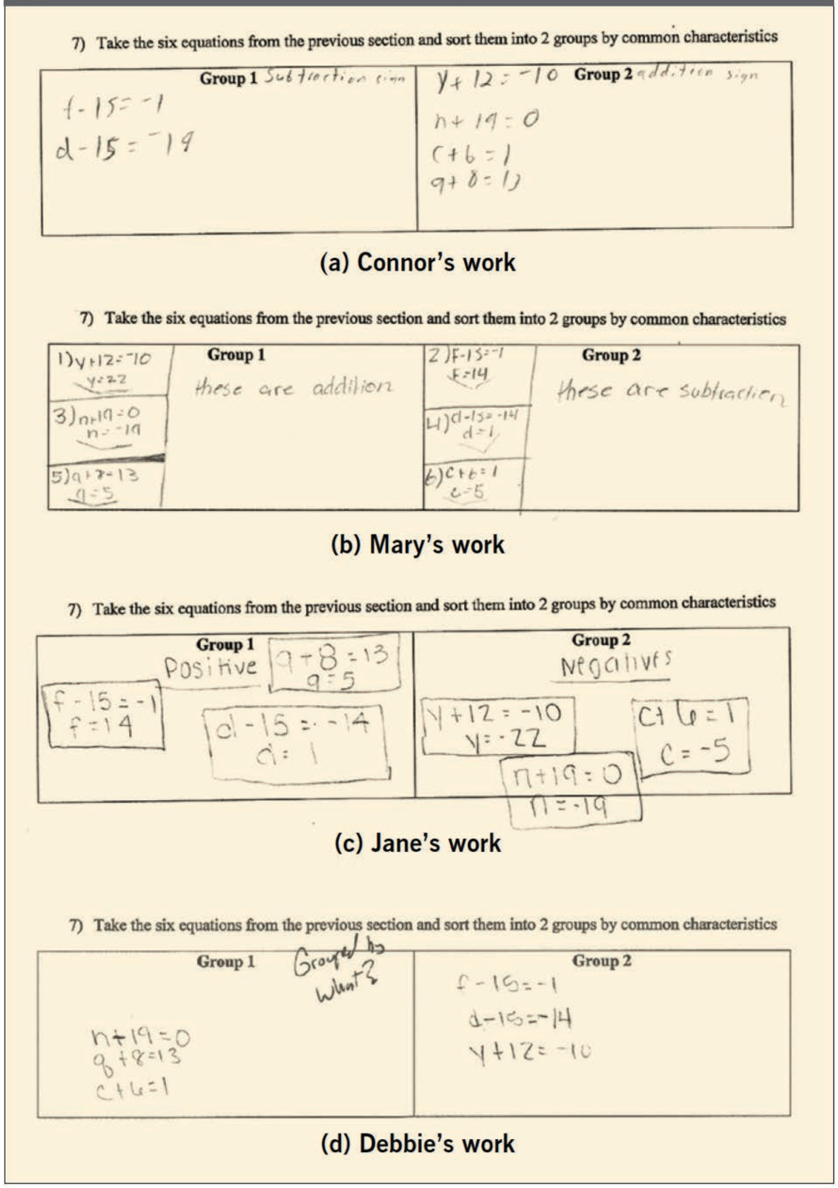
Fig. 5 Revisiting ideas often allowed for better class discussion and participation.



Students like Donna and Lois (see Figure 5) were improving on their answers and explanations from previous assignments. These two students—as well as others—were also more likely to volunteer their opinions in class when they had their explanations already on paper. This led to better class discussions in that more students engaged with the ideas. From my perspective, Find and Fix the Mistake problems made it clear what students did and did not understand in terms of mathematical procedures and concepts.

Problem sorts. Problem Sort questions asked students to sort equations they had already solved into two groups by using common characteristics, such as operations (shown by Connor and Mary's work in figs. 6a–6b), properties, nature of the solution (shown by Jane in Figure 6c), and so on. The open-ended nature of choosing the sorting criteria made this type of question challenging for students. Quite often I received answers like Debbie's (see Figure 6d), which appears to show a separation by operation but no explanation. This type of problem does not seem to exceed the cognitive level of the class, yet its usefulness in helping students better understand solving linear equations remains unclear to me.

Fig. 6 These students sorted equations by using common characteristics.



Create your own word problem. Students were asked to write a story that matched a given equation and expressions; basically, a word problem in reverse (an example is shown in Figure 7).

The level of demand for this task is much greater than those of previous assignments because students must—

- understand how the expressions relate to the equation;
- determine what the equation is asking; and
- make a meaningful story to go with the problem.

Unfortunately, the level of demand was evidently too high for students in the class.

Fig. 7 This grid illustrated a word problem in reverse.

2) The art teacher is making salt dough for an upcoming project. The ratio of flour to salt to water used to make salt dough is shown below.

Making Sour Dough	Story
Cups of flour: 2c	
Cups of salt: c	
Cups of water: ¾ c	
2c + c + ¾ c = 60 cups	

a. Write a story that matches the expressions and equation shown above.
b. Solve the equation. How many cups of each ingredient is the art teacher planning to use?

One determinant of how important or useful a problem was to students was whether they actually completed the problem. The Create Your Own Word Problem task was one of the most skipped problems in any of the homework assignments. Figure 8 shows that Kim and Connor gave answers that were nearly correct. They both solved the equation correctly but had difficulty in determining the number of cups needed for each ingredient. Kim's answers for each ingredient did not match the equation. Connor found a solution to the equation but not for each ingredient in the problem.

I did notice improvement for some students as they gained more experience with this type of problem. Figure 9 shows how Jane struggled with the first version of this problem and then a few weeks later how she improved with a different version. Improvement was evident for several students, and although the level of demand may have been high, the problem was useful in determining students' understanding and misconceptions of certain types of linear equations.

Fig. 8 These students had difficulty determining the number of cups needed for each ingredient.

Making Sour Dough	Story
Cups of flour: 2c	Mrs. Need is making salt dough.
Cups of salt: c	The ratio of the flour is 2c the
Cups of water: ¾ c	ratio of the salt is a amount of cups
2c + c + ¾ c = 60 cups	and the ratio of the water is 3/4 c.
2c + c + 3/4c = 60 cups	and the total ratio is 60cups.

a. Write a story that matches the expressions and equation shown above.
b. Solve the equation. How many cups ~~of~~ each ingredient is the art teacher planning to use?

$\frac{3.75c}{3.75} = \frac{60}{3.75}$ C=16cups Flour = 2c Water = 3/4 c Salt = 16c

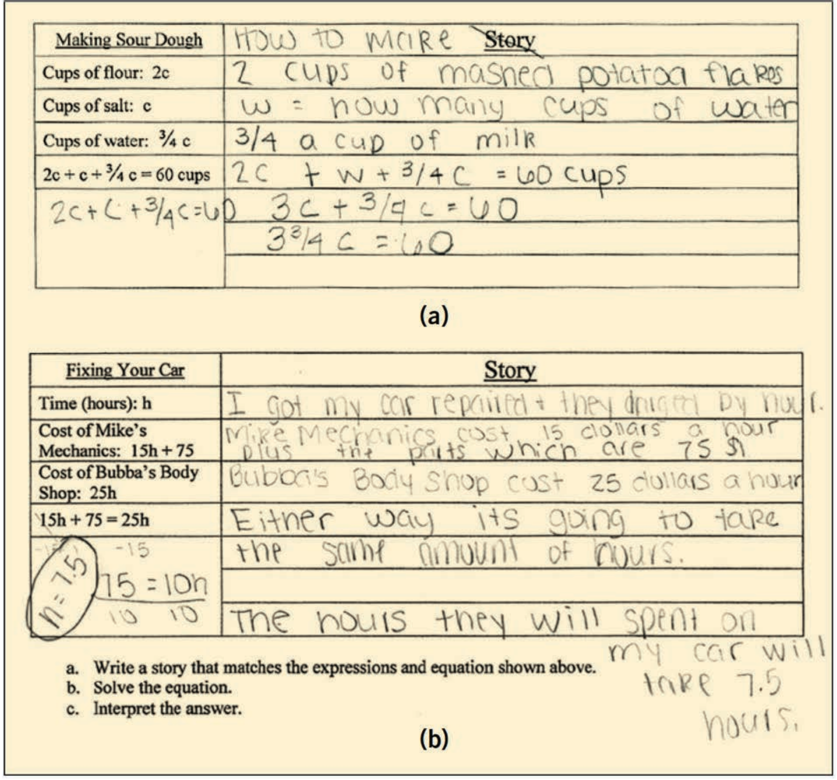
(a) Kim's work

Making Sour Dough	Story
Cups of flour: 2c	An art teacher is making special
Cups of salt: c	dough for a project. The ingredients are
Cups of water: ¾ c	two cups of flour, one cup of salt,
2c + c + ¾ c = 60 cups	and three quarters of water. Make an
$2c + c + \frac{3}{4}c = 60$cups	equation to find how many cups of
$3c + \frac{3}{4}c = 60$cups; $\frac{3.75c}{3.75} = \frac{60cups}{3.75}$	ingredients is the teacher is going to
C = 16cups	use.

(b) Connor's work

Justify your reasoning. This type of problem is designed to help students analyze why linear equations can have one, none, or infinitely many solutions. The concept of a math equation not having a solution was a new idea for many of the students at this grade level. This question was asked after students were already solving linear equations that might not have had a solution.

Fig. 9 Jane struggled with the first version of this problem. Weeks later, she improved with a different version.



Many students, including Sharon (see Figure 10a), repeated phrases that were discussed in class. Although these statements may be true, students did not offer any justification or evidence that they really understood how the inequality and the no-solution equation were related. Amy (see Figure 10b) also tried to relate the problem directly to class conversation. Like Sharon, Amy may simply be repeating what others have said in class, making it more difficult to perceive individual reasoning. Other students were able to provide some insight into how they viewed the problem. Kim attempted to describe actions she took to solve these types of problems (see Figure 10c). This does not necessarily demonstrate understanding; rather, it is an observed connection between the process and end result.

Connor's response (see Figure 10d) dug a little bit deeper. Connor was able to explain his answer by talking about how inequalities went with no solutions because the equations were "never true in the first place." He continued to describe how these equations may initially look like the others but do not hold up under mathematical investigation.

Students completed the Justify Your Reasoning problems relatively quickly, and most students attempted them. Allowing students to explain their responses in class helped them to make connections with other students and their ideas.

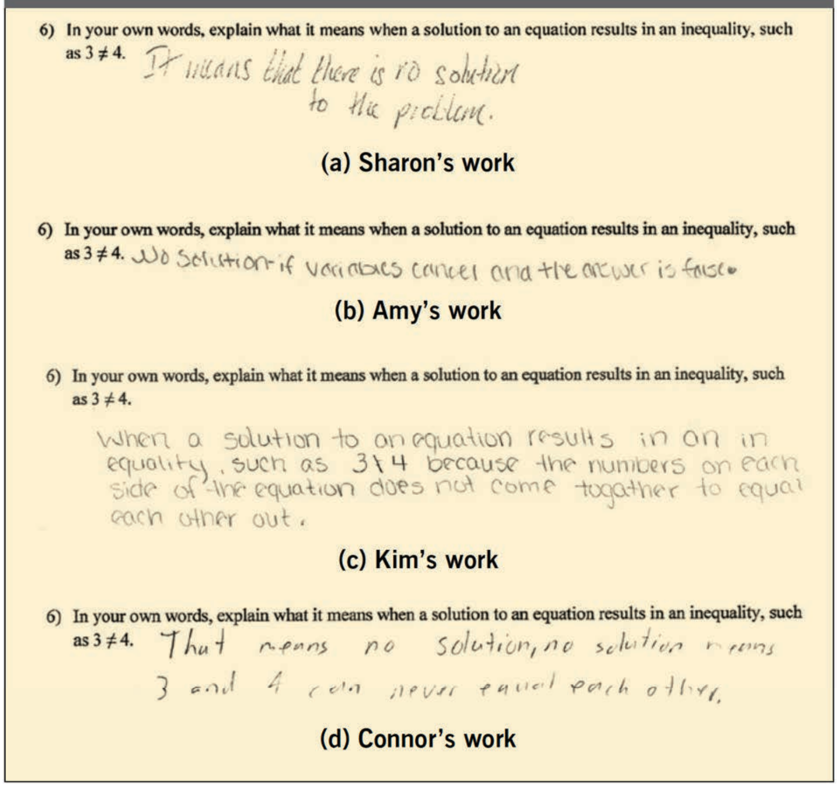
Positive Perception of Homework

I feel that these tasks had a positive impact on my students' perception of homework. A careful selection of homework tasks can help students practice, understand, and explore mathematical concepts. If students are confused about what a question is asking or how to begin, they are not as likely to persevere through the problem.

Teachers and students evaluate homework questions differently, and they cannot be labeled as simply easy or difficult. The appropriateness of the level of demand of the problem is important when considering what students should gain from completing a task.

Questions such as those in Find and Fix the Problem were very popular and successful with students. They had clear expectations, and students were able to determine answers that they thought made sense and were acceptable to them.

Fig. 10 Some students simply repeated classroom phrases, whereas others went a little deeper.



I found the benefits of using this style of homework to include the following:

- Improved class discussions: Students were able to explain the mathematical concepts with more confidence and use better vocabulary in the classroom setting.
- Teacher insights: Explanations for and justification of the homework problems made it easier for me to determine current levels of understanding as well as notice common misconceptions shared by the class.

This small change in my homework approach helped me gain information about students' perception of homework in a different way. Although the process did not determine why some students continually fail to complete their homework, it did help me see why certain students skip the occasional problem. In working with new homework practices, I learned how important it is for homework to contain an appropriate level of demand. Skills-practice problems are beneficial to students' math knowledge, but good cognitive problems require students to invoke deeper levels of thinking. I plan to continue to adjust my homework structure to include skills practice and cognitive thought. Problems with an appropriate level of demand and timely feedback allow students to learn from their homework and be confident that the work they do outside of class is meaningful.

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Changing the Way We Think About Mathematical Ability [1]

Caroline Junkins, University of Western Ontario

As a mathematician, many people I meet feel compelled to share with me their feelings toward math, good or bad. They tell stories of teachers, tests, or trigonometry, and how these experiences shaped their opinions and emotions tied to the subject. Over the years I've noticed two common themes in these stories. The first is that the storyteller provides an assessment of their mathematical ability, usually a variation of either "I was always good at math; it just comes naturally to me," or "I am just not a math person; I didn't get the math gene." The second is that regardless of that assessment, the storyteller will offer a statement along the lines of "The nice thing about math is that every question has a right and wrong answer; there's no ambiguity and there's only one way to do it."

> "'The nice thing about math is that every question has a right and wrong answer; there's no ambiguity.' Through my experience with mathematics I have learned that this statement is far from the truth."

Through my experience with mathematics I have learned that the latter statement is far from the truth. In an early calculus course, we are taught that the important part of an exam question is the process, not the final answer, and that there can be various successful approaches to the same question. In more advanced courses, we see that some questions may not have answers at all (for example "What is the sum of a divergent series?"), or that we may choose to work under the assumption that something is true, even if we are unable to verify its truth with proof (the Riemann Hypothesis for example). Even when answers do exist, they can be dependent on context, wording, or the set of agreed-upon axioms with which we are working (How do you feel about invoking the Axiom of Choice?).

From the viewpoint of a user rather than a creator of mathematics, these ambiguities—and the fascinating beauty they bring with them—are hidden. If one is told to use a specific tool for a specific task, without consideration of the why's and how's of it all, the task becomes routine and mindless. Many people exit the world of mathematics without making the transition from user to creator, and I believe this contributes to the false belief that every question in math has a right and wrong answer.

I started thinking about other judgments we make as a society about mathematics. If we could be so wrong about the black-and-whiteness of math, could we also be wrong about our understanding of mathematical ability? Is mathematical ability an innate attribute determined by genetics? Or is it part of ourselves that can be cultivated over time? This time, my personal involvement with math does not yield a clear answer.

There are areas of math towards which I feel more drawn, where the proof techniques seem more intuitive and the results more meaningful. I feel more confident and competent working in these areas, and success seems to come more easily. Does this mean that I'm just inherently more able in these areas, and less able in others?

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[1] Reprinted with the permission of the Canadian Mathematical Society, this article was originally published in CMS Notes, *Changing the way we think about mathematical ability*, C. Junkins, 2016, *48*(4), p. 11-12.

Last year I had the opportunity to teach a course in a subject which I despised as an undergraduate. When I first encountered the material, I found it pointless and incomprehensible. How was I expected to now teach this subject, and to convince my students that it was worth learning? Thankfully, this turned out to be a positive experience; when I returned to the material with a new perspective, it seemed completely different. Results and techniques which previously seemed arbitrary now made sense, and I was able to appreciate the importance of the subject to other fields.

"What if I had simply written myself off as less able in this subject and avoided it for the rest of my life? This certainly wouldn't have done me any good. Likewise, labelling our students as either low- or high-ability could limit their chances for success."

What if I had simply written myself off as less able in this subject and avoided it for the rest of my life? This certainly wouldn't have done me, or my students, any good. It seems to me that if mathematical ability is something we can mould over time then adopting an attitude of being "just not a math person" is incredibly unhelpful. Likewise, labelling our students as either low- or high-ability could limit their chances for success, or set them up for unnecessary failures.

So, which is it? Is mathematical ability determined at birth or something we can change with motivation and effort? As far as the psychological community is concerned, this is still a fiercely debated topic (see Kovas, Harlaar, Petrill, & Plomin 2005; Sigmundsson, Polman, Lorås, 2013). Luckily, however, our conversation about mathematical ability doesn't have to end here. It turns out that we can become better teachers and learners by simply believing this ability is malleable.

When it comes to our personal views of intelligence or ability, a socialcognitive approach pioneered by Carol Dweck (Dweck & Leggett, 1988; Dweck, 1999) classifies people as either entity theorists or incremental theorists. Entity theorists believe that one's ability is determined at a young age - or even at birth - and will not change over time. On the other hand, incremental theorists believe that ability is shaped over time and can be cultivated through effort.

Research by Dweck and her colleagues over various studies has consistently shown that both theories are held in roughly equal numbers throughout the population (Dweck, 1999). Furthermore, students from both groups demonstrate equal levels of overall academic performance (Licht & Dweck, 1984; Dweck, 1999). These theories of fixed ability (entity theory) or malleable ability (incremental theory) have been applied to various human traits such as personality, moral character, intelligence, and academic ability within a specific domain, such as language or mathematics. An individual can hold different theories for different traits, for example believing that personality is malleable but mathematical ability is fixed, or vice versa (Dweck, 1999).

Incremental theorists place higher value on learning goals, which reflect a desire to acquire new skills, master new tasks, or understand new things (Dweck, 1999). As students, incremental theorists are more likely to choose challenging problems, to persist after failure, and to develop strategies for improvement. As instructors, incremental theorists are more likely to offer effective feedback in response to both successes and failures, and to avoid making negative judgments of students from one or two poor test scores (Rattan, Good & Dweck, 2012).

It seems that in every way, holding an incremental theory of ability is an adaptive strategy which leads to success in both teaching and learning. So why do so many people, including successful students and instructors, seem to instead hold an entity theory when it comes to mathematics?

Dweck and her colleagues showed that both entity theorists and incremental theorists can have high or low self-esteem, as well as high or low confidence in their abilities (Dweck, 1999). The difference is that for entity theorists, these feelings are tied to performance: to them, a grade is not just a reflection of their current effort or skill, but a judgment of their overall intelligence and potential. If an entity theorist receives a low grade on a particular test, they tend to feel devastated, hopeless, and depressed. In the future, they are more likely to shy away from challenging problems in that subject, or avoid the subject all together (Dweck, 1999). In a society where math anxiety is unfortunately so common, the entity theorist's reaction is an accepted one and provides an easy way to rationalize failures and avoid them in the future.

On the other hand, when an entity theorist receives a good grade, or praise for their intelligence, this validates and strengthens the belief that they are "good at math." Holding this belief allows the student to approach new problems with an expectation of success, and may give them a feeling of superiority over those who struggle with math. As long as the student keeps succeeding, their belief is never challenged and their self-confidence will continue to grow. An entity theory may give one a sense of security and stability (Dweck, 1999), and can be very attractive since the negative consequences of holding such a theory are only visible in the face of failure.

> "Failure is an inevitable part of learning, and it is in our best interest to be as prepared as possible for when it occurs, either to us or to our students."

Failure, however, is an inevitable part of learning, and it is in our best interest to be as prepared as possible for when it occurs, either to us or to our students. By believing in an incremental theory of intelligence, or at least by reacting and proceeding as if we did, we can mitigate the negative emotions which accompany failure and create an opportunity for growth. Within an incremental framework, a failure can be acknowledged as a sign that increased effort or new learning strategies are required (Dweck, 1999). The student and instructor can work together to identify deficiencies and develop a plan for addressing them. A key benefit of working within an incremental framework is that we have a better chance of separating feelings of identity and self-worth from current performance.

If our society were to hold the view that mathematical ability is something that grows along with us, it is possible that more people could be encouraged to make the transition from user to creator of math. At the very least, the conversations we have about math could involve more than a simple declaration of one's own perceived ability.

To the person who tells me that math "just comes naturally." I could present challenging ideas or problems to expand their understanding without damaging their self-esteem; to the person who "didn't get the math gene," I could offer suggestions for reviewing concepts or studying more effectively without labelling them as low-ability. Math itself is certainly not a fixed entity, and our relationship with it should not be either.

From Pictograms to Solving Linear Systems

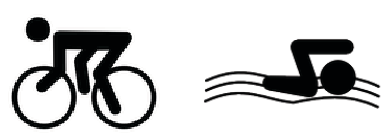
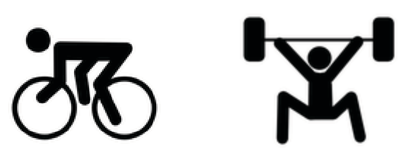
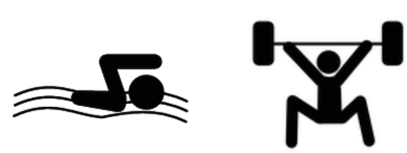
Carly Ziniuk

Last year, I introduced my Grade 9 students to the concept of linear systems graphically. They did enjoy finding intersection points using technology and even by graphing, but then when we transitioned to expressing these algebraically, they were struggling with the loss of a visual representation.

It was near the end of the year, and the scenario of two different pizza companies and their differing topping costs just did not seem to interest my students. They just couldn't get their heads around how to add or subtract the resulting equations and why that made any sense, given that they started from a graphical representation, as the Ontario curriculum recommends for Grade 9. They were frustrated, however, by a return to algebra tiles, which they saw as baby-ish, but I knew they were visually strong and liked concrete manipulation.

I started looking at some problems I had previously used with junior and middle school students, which involved visual representations in a chart. Because we were getting closer to the summer and the 2016 Summer Olympics and Paralympics in Rio, I thought it would be good to extend these kinds of problems using sport pictograms. Here's an example, followed by some examples of student reasoning:

1. Figure out how much each Olympic sport is worth.

	**24**
	**14**
	**32**

Alessa's solution shows:

- "If I add them all together, I get 2 of each and then 24 + 14 + 32 = 70."

Lauren continues:

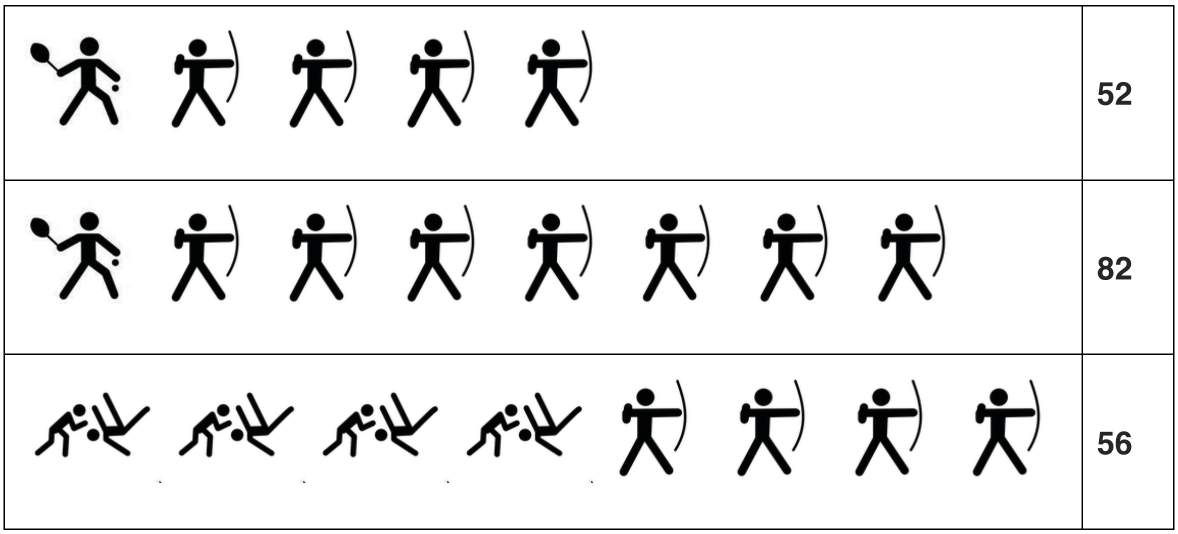
- "One each then would be 35."

- “If you look at the last one, I can tell the missing one, the biker must be 3 because that is what is left from the total 35.”

Alessa comes back with,

- “So that means the biker is 3, the weightlifter is 11, and the swimmer is 21.”

2. Figure out how much each Olympic sport is worth.



Right away, Alessa writes:

- 3A = 30 (and she circles the three archers at the end of the second line!)
- So A = 10

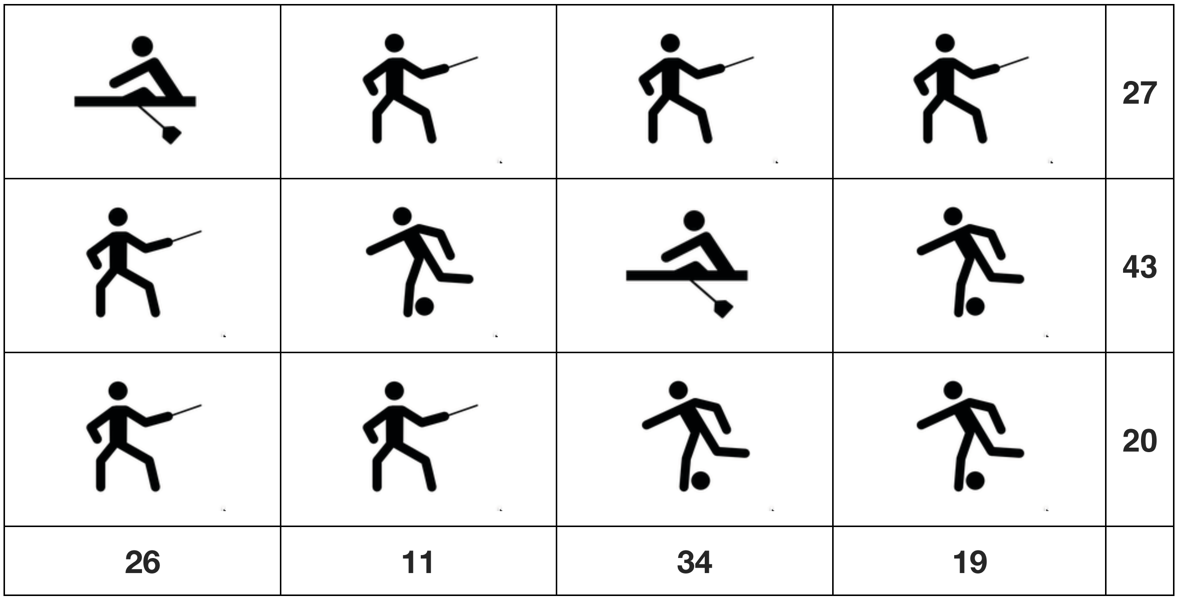
And right away, the rest of the students start chiming in with the values of the other pictograms, without using variables.

I found it easy with both of these examples to then move to showing students that what Alessa was doing with the first two rows, which can be represented as

$$1T + 4A = 52$$
$$1T + 7A = 82,$$

can be expressed algebraically as subtracting one equation from the other because the coefficients of A matched, and that what followed was implicit substitution.

While the students were working through these problems, they solved them in so many different ways that when we moved to a more complicated look at linear systems, it made sense to them why we needed to add or subtract the given equations. They were able to write equations that “matched” pictograms like these and started subtracting the resulting equations:

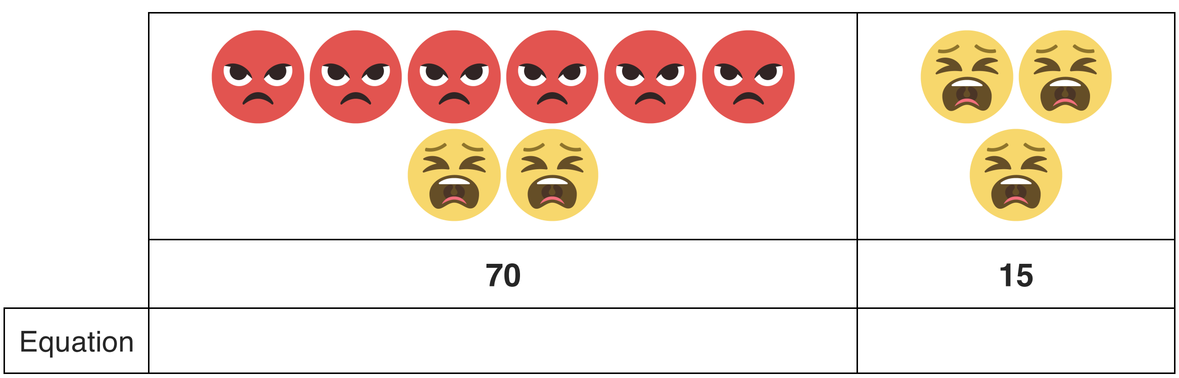


Right away, one of the students noticed the similarities between the first row and the first column, and wrote

> 1R + 3F = 27
> 1R + 2F = 26
> Therefore, 1F = 1 and 1R = 24.

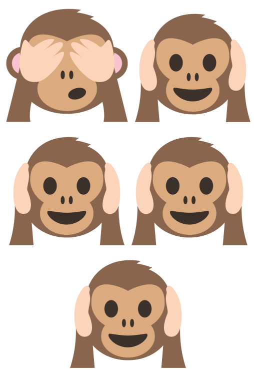
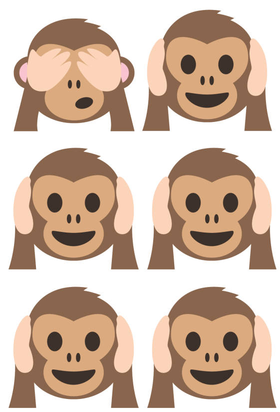
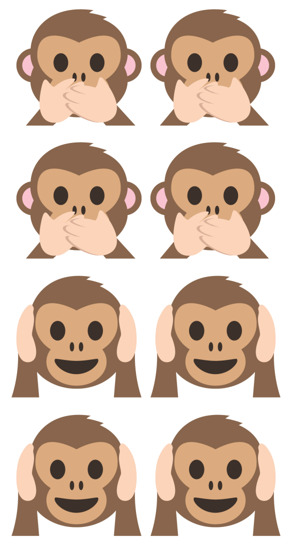
This year, I decided I was going to work with these problems earlier in the course, during our introduction to algebra. I looked at some simpler problems, even just to give students the opportunity to practice defining variables and setting up some simple equations to solve. I thought they would have some fun with emojis in the same way that my previous students did with sports pictograms (emojis provided free by EmojiOne).

Here is an example of the kinds of problems we started with:



I found that by first having students write less complicated equations, including ones with only one variable, they were able to do substitution more easily and more consistently later on for more complicated systems, such as these:

			
	24	**14**	**32**
Equation			

			
	52	**82**	**56**
Equation			

These tasks gave me a lot of opportunities to see how the students solved problems, but also a simple way to make sure that they were able to properly identify variables and create equations that corresponded to a given situation.

The final part of this lesson was to have the students make their own problems for each other in the form of a quiz. The problems had to include at least three equations and at least three emojis. Many of the students used a chart like the ones below, since they are really easy to set up (but much harder to solve!). It was really interesting to see the difference between their thinking and their communication using algebra. After I had already assigned this "making of a quiz" for homework, I was very pleasantly surprised to see some of the students using negatives and decimals! Great fun, and a bit of frustration, ensued. Without me having to say a word, it was clear to the students why using algebra was so much easier. Now, I can't wait to get to linear systems with these students at the end of the year! (Also, I now have a full class set of different quizzes to use for the future, and answers, too!)

Lauren's Algebra Quiz: Determine the "costs" of these items using equations and fully explained!

👽	👽	😜	🐰	**55**
😜	👽	🐰	😜	**54**
😜	😜	😜	👽	**33**
25	**26**	**45**	**46**	

Alessa's Answers: The first column is 1A+2S = 25. The last row is 3S + 1A = 33. The difference is that the row and the column has a 1S difference. And that is 8 total. Which means a Smiley Tongue is 8. Then I can figure out that A=9 (Alien) and R = 29 (Rabbit).

If you are looking for more examples of these kinds of problems, head to the Figure This: Math Challenges for Families website by NCTM, taking a look in particular at the problems Gone Fishin' (Problem 58) and Smiles (Problem 30).

I hope you and your students enjoy these kinds of problems as much as we did!

Carly Ziniuk, carlyziniuk@gmail.com, teaches Grade 9 Mathematics, Grade 12 Data Management, and AP Statistics at The Bishop Strachan School in Toronto, Ontario, Canada. She also works with more than 20 Junior School students with a weekly Math Olympics club and a week long Summer Enrichment program for Middle School students. She is interested in using real-life data to engage her students in solving problems.

In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description. If you know about an event that should be on our list, please contact us at thevariable@smts.ca.

Within Saskatchewan

Workshops

Using Tasks in High School Mathematics
November 20, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Using tasks in a high school mathematics classroom can provide rich opportunities for differentiated learning and authentic assessment. How do we choose tasks that meet both curricular outcomes and student needs? Tasks allow students to enter mathematics where they are at and extend their learning. In this workshop we will look at a variety of resources for finding good high school tasks. We will also reflect and discuss what planning and teaching moves can assist in maximizing student learning through mathematics tasks.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/using-tasks-high-school-mathematics-0

Using Tasks in Middle Years Mathematics
January 15, 2018, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Using tasks in a middle years mathematics classroom can provide rich opportunities for differentiated learning and authentic assessment. How do we choose tasks that meet both curricular outcomes and student needs? Tasks allow students to enter mathematics where they are at and extend their learning. In this workshop we will look at a variety of resources for finding good middle years tasks. We will also reflect and discuss what planning and teaching moves can assist in maximizing student learning through mathematics tasks.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/using-tasks-middle-years-mathematics-1

Early Learning with Block Play—Numeracy, Literacy, and So Much More!
January 26, 2018, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

This is a one-day workshop for early learning educators from prekindergarten, kindergarten and Grade 1 to work collaboratively to discover and deepen their understandings around the many foundational skills that children develop during block play. Through concrete, hands-on activities, participants will experience and examine the many connections between block play and curricular outcomes, and the current research on the topic. Participants will have opportunity for reflection on their current practice, planning for block play and for creating a network of support.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/early-learning-block-play-%E2%80%93-numeracy-0

Conferences

Saskatchewan Understands Math (SUM) Conference
October 23-34, Saskatoon, SK
Presented by the Saskatchewan Mathematics Teachers' Society (SMTS), the Saskatchewan Educational Leadership Unit (SELU), and the Saskatchewan Professional Development Unit (SPDU)

This year, the Saskatchewan Mathematics Teachers' Society, the Saskatchewan Educational Leadership Unit and the Saskatchewan Professional Development Unit are partnering to co-ordinate a province-wide conference to explore and exchange ideas and practices about the teaching and learning of mathematics. The Saskatchewan Understands Math (SUM) conference is for mathematics educators teaching in Grades K-12 and all levels of educational leadership who support curriculum, instruction, number sense, problem-solving, culturally responsive teaching, and technology integration, and will bring together international and local facilitators to work in meaningful ways with participants in a variety of formats. This year, SUM is featuring keynote speakers Steve Leinwand of the American Institutes for Research and Lisa Lunney-Borden of St. Francis Xavier University. See the poster on page 4, and head to our website (www.smts.ca) for more information.

Beyond Saskatchewan

MCATA Fall Conference 2017: A Prime Year for Mathematics
October 20-21, 2017, Enoch, AB
Presented by the Mathematics Council of the Alberta Teachers' Association

Join the Mathematics Council of the Albeta Teachers' Association in celebrating their annual fall conference in Enoch, Alberta. This year's keynote speakers are Michael Pruner, a high school mathematics teacher with a Thinking Classroom in Vancouver and president of the BC Association of Mathematics Teachers, and Sunil Singh, author of the book, *Pi of Life: The Hidden Happiness of Mathematics* and self- proclaimed Mathematical Jester who is transforming the way mathematics is revealed and discussed in North America.

See http://www.mathteachers.ab.ca/information-and-registration.html

NCTM Annual Meeting and Exposition
April 25-28, 2018, Washington, DC
Presented by the National Council of Teachers of Mathematics

Join more than 9,000 of your mathematics education peers at the premier math education event of the year! NCTM's Annual Meeting & Exposition is a great opportunity to expand both your local and national networks and can help you find the information you need to help prepare your pre-K–Grade 12 students for college and career success. Classroom teachers, administrators, math coaches, supervisors, college professors, and preservice teachers can all benefit from the sessions and learning at this event. Improve your knowledge and skills with high-quality professional development and hands on activities; gain insights by connecting and sharing with like-minded educators; collect free activities that will keep students engaged and excited to learn; and learn from industry leaders and test the latest educational resources.

See http://www.nctm.org/Conferences-and-Professional-Development/Annual-Meeting-and-Exposition/

Online Workshops

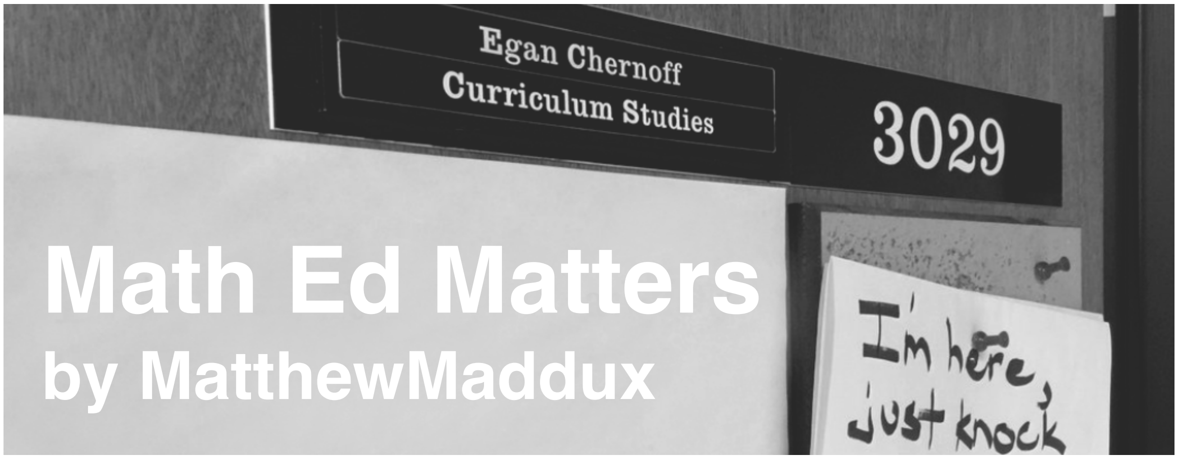
Education Week Math Webinars
Presented by Education Week

Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling and Differentiation.

Past webinars: http://www.edweek.org/ew/webinars/math-webinars.html
Upcoming webinars:
http://www.edweek.org/ew/marketplace/webinars/webinars.html

Did you know that the Saskatchewan Mathematics Teachers' Society is a **National Council of Teachers of Mathematics Affiliate**? When registering for an NCTM membership, be sure to support the SMTS by noting your affiliation during registration.

NCTM AFFILIATE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS



Math Ed Matters by MatthewMaddux *is a bimonthly column telling slightly bent, untold, true stories of mathematics teaching and learning.*

Abhorrent Mathematical Algorithms: Mathematical Abhorithms

Egan J Chernoff

Mathematical algorithms—that is, sets of rules or step-by-step procedures used to solve a mathematics problem—are a staple of mathematics classrooms around the world. The reasoning is simple: Following the steps you were taught (e.g., for long division) will, if all goes well, lead you to the correct answer for a corresponding problem (which is, of course, very important in math class). Let me be clear from the outset: I have no issues with mathematical algorithms. I do, however, take issue with what I call *mathematical abhorithms*.

A mathematical abhorithm is the name I give to an abhorrent mathematical algorithm. I deem an algorithm abhorrent if the set of rules used to correctly solve a mathematics problem—that is, the mathematical algorithm—either (1) has no mathematical basis, (2) ignores any underlying mathematical basis, or if (3) the link between the abhorithm and any mathematical basis is not adequately taught. Much like mathematical algorithms, mathematical abhorithms are, unfortunately, epidemic in mathematics classrooms around the world. And, far from being harmless, they represent a huge missed opportunity in the teaching and learning of mathematics. Let's take a look at a few examples.

Converting from Decimal to Percent

"To convert from a decimal to a percent, you move the decimal two places to the right, then write the percent symbol." This, I contend, is a mathematical abhorithm. Every year, I stand in rooms full of future elementary, middle, and high school math teachers and ask them: "To convert from a decimal to a percent you move the decimal two places to the right, then write the percent symbol—right?!" The room is always quiet. After all, until that point, many of the future math teachers have likely converted from decimal to percent all their lives by moving the decimal two places to the right, then writing the percent symbol, with little or no issues whatsoever. Maybe you have, too. So before reading on, ask yourself: To convert from a decimal to a percent, you move the decimal two places to the right, then write the percent symbol—right?!

Well, yes and no. Yes, in the sense that you get the right answer. Nevertheless, the algorithm is abhorrent. Why should decimals "move"? Why should you write down the percent symbol? Those of you who are a step ahead of me with an explanation to make the example less abhorrent might only be making matters worse. In particular, if you're thinking that the *reason* you move the decimal two places to the right, then write the percent symbol is because "you multiply by 100," then, I contend, we definitely have a mathematical abhorithm on our hands. Why?

Here's why[1]. If you take a decimal, say 0.37, and multiply it by 100, the act of multiplying by 100 changes the original number. No big deal, right? Wrong. Multiplication, as a gross oversimplification, can be understood as scaling one number by a given factor. This is why 0.37 turns into 37 when you multiply it by 100: it has been scaled by a factor of 100. However, when converting from decimal to percent, what we are trying to achieve is not scaling, but rather preservation of the number (0.37) with a change in the way it is represented—in this case, to a percent, that is, out of 100.

Here is one way we could make the change. We multiply the number (in this case, 0.37) by 1 (because 1 is the only number by which you can multiply without changing the original number). But when we do so, we multiply by 1 "disguised" as 100/100 ($0.37 \times 100/100$), which results in $(0.37\times100)/100$, which is equal to 37/100 and, finally, as we are told, you can exchange "something out of 100" by appending the percent symbol to the numerator (resulting in 37%). I think I know what you're thinking at this point: "What's the big deal?"

Let's call the above process *Scenario 1*: to repeat, we multiply 0.37 by 1 (knowing that multiplying by 1 will not change the original number), expressed as 0.37×1, but we choose to "disguise" 1 as 100/100 ($100/100 = 1$, even though it may not look like 1), which equals $(0.37 \times 100)/100$, which (after some calculation) is equivalent to 37/100, which is now an expression that can be interpreted as "out of 100," which means that we can use the percent symbol, which literally means "out of 100" and, finally, we get the answer: 37%. Now let's consider what we will call *Scenario 2*: Move the decimal to places to the right, then add the % symbol.

> "This algorithm is abhorrent because it is not mathematically sound. But why, then, would an abhorithm ever take the place of an algorithm?"

Scenario 1 represents a school mathematics algorithm. Scenario 2, I contend, represents a school mathematics abhorithm. In Scenario 2, the algorithm—that is, the set of rules used to correctly solve the mathematics problem—is abhorrent because it is not mathematically sound. But why, then, you might be asking, would an abhorithm ever take the place of an algorithm? Good question!

Imagine, for a moment, the following conversation between a teacher and a student immediately after their math class has just been presented with Scenario 1:

> *Student*: "Excuse me, Sir, it looks like you can just take the decimal move it two places to the right, write the percent symbol, and get the right answer. Is this true?

[1] For those of you who are chomping at the bit to find flaw in my impending explanation, I make no guarantees that it is bulletproof. I ask, however, that you try to see the forest for the trees here. Me, I can sleep at night knowing that my way is less abhorrent than "move the decimal two places to the right, then write the percent symbol."

Teacher: "Yes, but…"
Student: "And if I just take the decimal move it two places to the right and write the percent symbol, will I get every question right on all my homework before next class?"
Teacher: "Yes, but…"
Student: "Awesome, and if I just take the decimal move it two places to the right and write the percent symbol, will I get every question right on the quiz we're gonna have on this topic?"
Teacher: "Yes, but…"
Student: "Wow! Ok, and if I just take the decimal move it two places to the right and write the percent symbol, will I get every question right on our upcoming chapter test?"
Teacher: "Yes, but…"
Student: "Sir, one last question. If I just take the decimal move it two places to the right, write the percent symbol, and get every question right on the homework and on the quiz and on the test, I'll get a pretty high mark in math, right?"
Teacher: "Yes, but…"
Student: "Sir, if I get a pretty high mark in math, then people like my parents will think I'm pretty good at math, right?"
Teacher: "Yes, but…"
Student: "Sir, actually, one final question, why would I bother with your example [Scenario 1] when I can just convert from a decimal to a percent by moving the decimal two places to the right and writing the percent symbol?"

Another good question! And, if you consider the situation from the student's perspective, it's hard to see why, exactly, you shouldn't just move the decimal two places to the right and write the percent symbol, knowing full well that this abhorithm will get you the right answer. Every. Single. Time.

But wait! There's more.

Adding and Subtracting Fractions
The "bow tie" method for adding and subtracting fractions is, I contend, another mathematical abhorithm. To demonstrate the bow tie method, consider the addition of 1/2 and 1/3. According to this particular abhorithm:

Step 1: Multiply the two numbers on the bottom (2×3) and write that number on the bottom;
Step 2: Multiply the two numbers that lie on the 45-degree angle that starts on the bottom right and moves towards the top left of the page (3×1), and write that number on the top;
Step 3: Multiply the two numbers that lie on the 45-degree angle that starts on the bottom left and moves towards the top right of the page (2×1), and write that number on the top;
Step 4: Add the two top numbers together (3+2/6 = 5/6);
Step 5[2]: Enjoy the fruits of your labour.

See Figure 1 for an illustration.

[2] In the interest of full disclosure, I added Step 5.

$\frac{1}{2}+\frac{1}{3}$

Step 1: Multiply the bottom two numbers, which will be the denominator

$\frac{1}{2}+\frac{1}{3}$

Step 2: Multiply the numbers along the first diagonal

$\frac{1}{2}+\frac{1}{3}$

Step 3: Multiply the numbers along the second diagonal

$\frac{1}{2}+\frac{1}{3}=\frac{3+2}{6}=\frac{5}{6}$

Step 4: Add the previous two numbers together and write the result in the numerator

Figure 1: The "bow tie" method for adding fractions

Voilà: the correct answer. Every. Single. Time. Sure, the answer will not necessarily be in "lowest terms" (a convention that also tends not to be fully justified for students in math class), but barring any arithmetic errors, the bow tie method will always work.

The reason that I contend the bow tie method is abhorrent is because the action of multiplying numbers along a horizontal line, followed by multiplying numbers along 45-degree angles, is not *why* you get the right answer. However, lots and lots of students *do* think that multiplication along some particular angle is "why" 1/2 + 1/3 = 5/6. The bow tie method is also abhorrent because it represents a missed opportunity for studying a topic (fractions) that, for many people, was a stumbling block to pursuing further school mathematics. So let's take a minute and look at the missed opportunity associated with the bow tie method.

> "Lots and lots of students *do* think that multiplication along some particular angle is "why" 1/2 + 1/3 = 5/6."

As with the previous example of converting from a decimal to a percent, let's look at two possible scenarios that could be presented to students who are learning to add fractions. For the sake of consistency, we'll call the "bow tie" method *Scenario 2*. In *Scenario 1*, on the other hand, we will try to move from abhorithm to algorithm. If you remember adding fractions from your days in school, then you probably remember the following phrase: "You can't add halves and thirds." I am not particularly fond of this phrase, because you certainly *can* add halves and thirds; however, you first need to change the way in which these numbers are represented. Here, we return to the "1 in disguise" notion that was presented earlier in the case of converting from a decimal to a percent. In adding 1/2 and 1/3, we want to preserve the numbers that we are starting with, but change the way in which they are represented. To do so, we take the number 1/2 and multiply it by 1 (because doing so won't change the number), but not by 1 as 1, but rather 1 "disguised" as 3/3. So, we get 1/2 × 3/3 = 3/6 (yes, yes, I know—using the abhorithm[3] "top times top and bottom times bottom"). Similarly, we take the number 1/3 and multiply it by 1 (because doing so won't change the number), but we choose a different disguise for 1, which in this case is 2/2. Now, 1/2 + 1/3 is represented as 3/6 + 2/6, which "allows" us to add the two numerators together because we have a common denominator. We thus get the answer, 5/6.

Let's consider a similar conversation between a student and a teacher after the class has just been presented with Scenario 1:

[3] Utilizing the multiplication of fractions in the addition and subtraction of fractions, even though in most textbooks and curricula the topic of fraction multiplication is presented after the addition of fractions, is another issue for another time.

Teacher: "So, class, are you comfortable with the addition of fractions?"
Student: "Yes, but… Sir, it appears that if I just use this bow tie method, I'll get the right answer!"
Teacher: "Yes, but…"
Student: "Sir, if I use the bow tie method, then I will get every question right on the homework and the quiz and the test, which means I'll get a pretty good mark in math, right?"
Teacher: "Well, yes, that's true, but you won't really understand what's really going on when you add fractions. You won't be able to explain *why* the bow tie method works."
Student: "Yeah, but I'll still get the right answer every time, right?"
Teacher: "Yes, but…"

Despite the fact that it produces correct results (Every. Single. Time.), the bow tie method is a mathematical abhorithm, just like the method of moving the decimal two places to the right (because you're multiplying by 100), then writing the percent symbol to convert from a decimal to a percent. When considered together, the two abhorithms start to snowball. However, they also represent a huge missed opportunity in the teaching and learning of mathematics.

House of Cards
Consider, again, a student's perspective on the two abhorithms. There are two disjoint procedures that need to be remembered: In the first example, a student has to remember to move the decimal two places to the right and write down the percent symbol, and in the second example, to implement the bow tie method. Now, here's the missed opportunity: At its core, the underlying notion associated with the two examples is one and the same—that is, the key to both the bow tie method and the method of moving the decimal, then writing the percent symbol when converting from decimal to percent is a distinction between number and numeral. In other words, both examples are asking to have the *numbers* in question preserved while their representation is changed, which requires using different *numerals*. In both examples, this is achieved by multiplying by the number 1 under different disguises. When the disguises are chosen properly, the rest of the question takes care of itself.

> "Having to remember two things instead of one isn't really that big of a deal. But what if you were bombarded with mathematical abhorithms—lots of them?"

In keeping with the theme, let's look at things in terms of two different scenarios. In Scenario 1, a student has a grasp of the underlying concept for the two distinct, yet related examples (the distinction between number and numeral). In Scenario 2, the student does not make the underlying connection between the two different examples and, in order to answer corresponding questions, needs to recall two distinct mathematical abhorithms. Alternatively stated, in Scenario 2, the student is not aware, or does not care about the underlying concept that underlies the two problems.

Of course, having to remember two things instead of one isn't really that big of a deal. But what if, in the mathematics classroom, you were bombarded with mathematical abhorithms—lots of them? For example:

- *To convert from decimal to percent, move the decimal two places to the right and write the percent symbol*
- *To add fractions, use the bow tie method*
- *You can't subtract a bigger number from a smaller number*
- *Reasons for which we shall not discuss, negative times negative equals a plus*
- *Why ask why? Just invert and multiply*
- *When you multiply by ten, add a zero*
- *When you multiply by one hundred, add two zeros*
- *When you drag a number across the equals sign, you turn a positive into a negative*
- *To complete the square, you add 35 and subtract 35 from the equation*
- *To solve an inverse function, swap y and x, solve for x, then swap back*
- *[Insert your favourite mathematical abhorithm here]*
- *[And here]*
- *[And here]*
- *[And here]*

Each mathematical abhorithm that a student encounters is another distinct set of "rules" that they must become familiar with and remember. How many mathematical abhorithms could one deal with, realistically, before things became difficult to manage? Let me phrase that a little bit differently. Let's say that each mathematical abhorithm represents a single card in a house of cards. How big do you think your house of cards could get before it came crashing down?

> "Let's say that each mathematical abhorithm represents a single card in a house of cards. How big do you think your house of cards could get before it came crashing down?"

Unfortunately, for many students, this ever-growing house of cards represents their school mathematics experience. Students are tasked with building bigger and bigger houses of cards, but, of course, the foundation of these houses is not as sturdy as one would like. Yes, some people's houses get bigger than others', for various reasons. Some are simply able to remember more mathematical abhorithms—that is, some are better at following orders than others. Perhaps some are able to establish conceptual connections that are not presented, or inadequately presented. However, most of these houses of cards will eventually come crashing down. This particular point in time is, perhaps (and if so, to nobody's surprise) when the individual develops a distaste for mathematics.

Most people that I've asked have a distaste for either fractions or algebra, or both. And when I ask at which particular point mathematics started to overwhelm them, they typically respond—you guessed it—"fractions" (the *other* other f-word) or "algebra." In some cases, though, the house of cards comes crashing down much later. For those who are able to build a house of cards big enough to get through high school, their house of cards usually comes crashing down during their first-year Calculus class in university. Imagine sitting in the back of a room with 300 other first-year calculus students, barely able to see the professor at the front of the room, and slowly realizing to yourself: "Uh oh!!" What a horrible feeling this must be: All those different abhorithms, at one point manageable, suddenly become unmanageable. As they say, the bigger they are, the harder they fall.

I was lucky: My own house of cards came crashing down—oh yes, it did come crashing down ("*Prove* that angle-side-angle proves congruency?!")—much later than those of many others. Looking back, though, I can relate; I completely understand why some houses collapse earlier. Consider the following phrase: "The first rule of dividing fractions is you never divide fractions." To reiterate: The *first rule of dividing fractions* is you *never divide fractions*. Any discerning student should be questioning things at this point. But wait, there's more to the statement: "The first rule of dividing fractions is you never divide fractions. Instead, you flip the second number—only the second number!—and then multiply the resulting fractions to get the answer." By now, students should have a number of questions that they want to ask… shouldn't they?! For example: "Why do we flip the second number and not the first?" Once again, though, we need to take the student's perspective into account.

> "'The *first rule of dividing fractions* is you *never divide fractions.*' Any discerning student should be questioning things at this point.'"

From a student's perspective, it makes sense to not ask why, to "just invert and multiply"—armed with this abhorithm, they will likely get every question right on every homework assignment and quiz and test related to fraction division that they will ever be given. As a result, they will get a good mark in math—and people will even think that they are good at it! No harm done, perhaps; after all, students are less often presented with scenarios where they truly need to know "why." If we truly want to honour the students in our math classrooms, then, we have to realize that from their perspective, it makes good sense to try to keep track of all of the mathematical abhorithms that they are taught—even though this will result in a house of cards that will, eventually, come crashing down.

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Saskatchewan Mathematics Teachers' Society