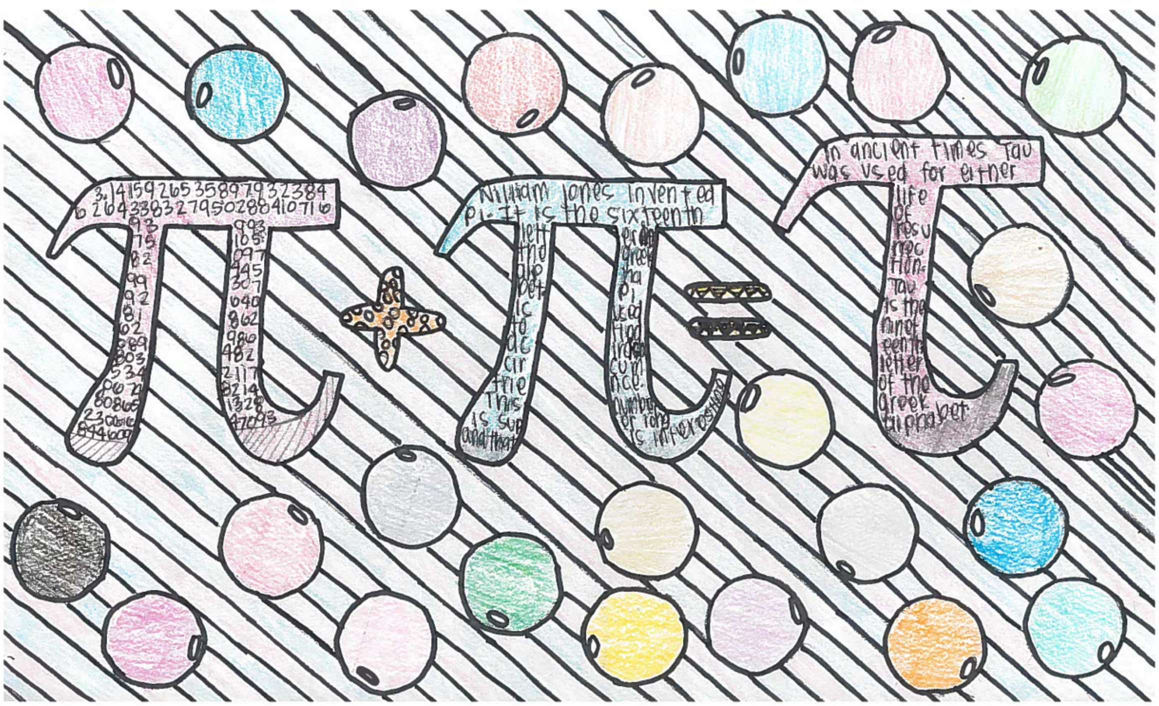
# The Variable

Presented by the
Saskatchewan Mathematics Teachers' Society

Volume 1 | Issue 3 | June 2016


**Encouraging mathematical habits of mind:**
**Puzzles and games for the classroom**
Domino puzzles and Rectangles
*Susan Milner*, p. 22




**Spotlight on the Profession**
*In conversation with Dr. Gale Russell*, p. 12

**Negative zero**
*Cindy Smith*, p. 19

**Tommy Douglas Math Fair '16**
*Heidi Neufeld*, p. 6; *Diana Naumova*, p. 10

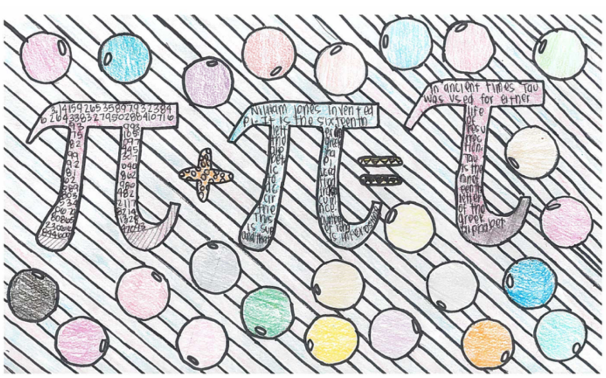
**Summer "PD" – For kids!**
*Sharon Harvey*, p. 34

Saskatchewan Mathematics Teachers' Society

## In this issue



## Cover art



"This piece was created by one of my Grade 9 students. The students were to research a mathematically significant number and to create a piece of art that represented it."

*Sharon Harvey, Saskatoon Public Schools*

## SMTS Executive (2016-2018)



## *The Variable*



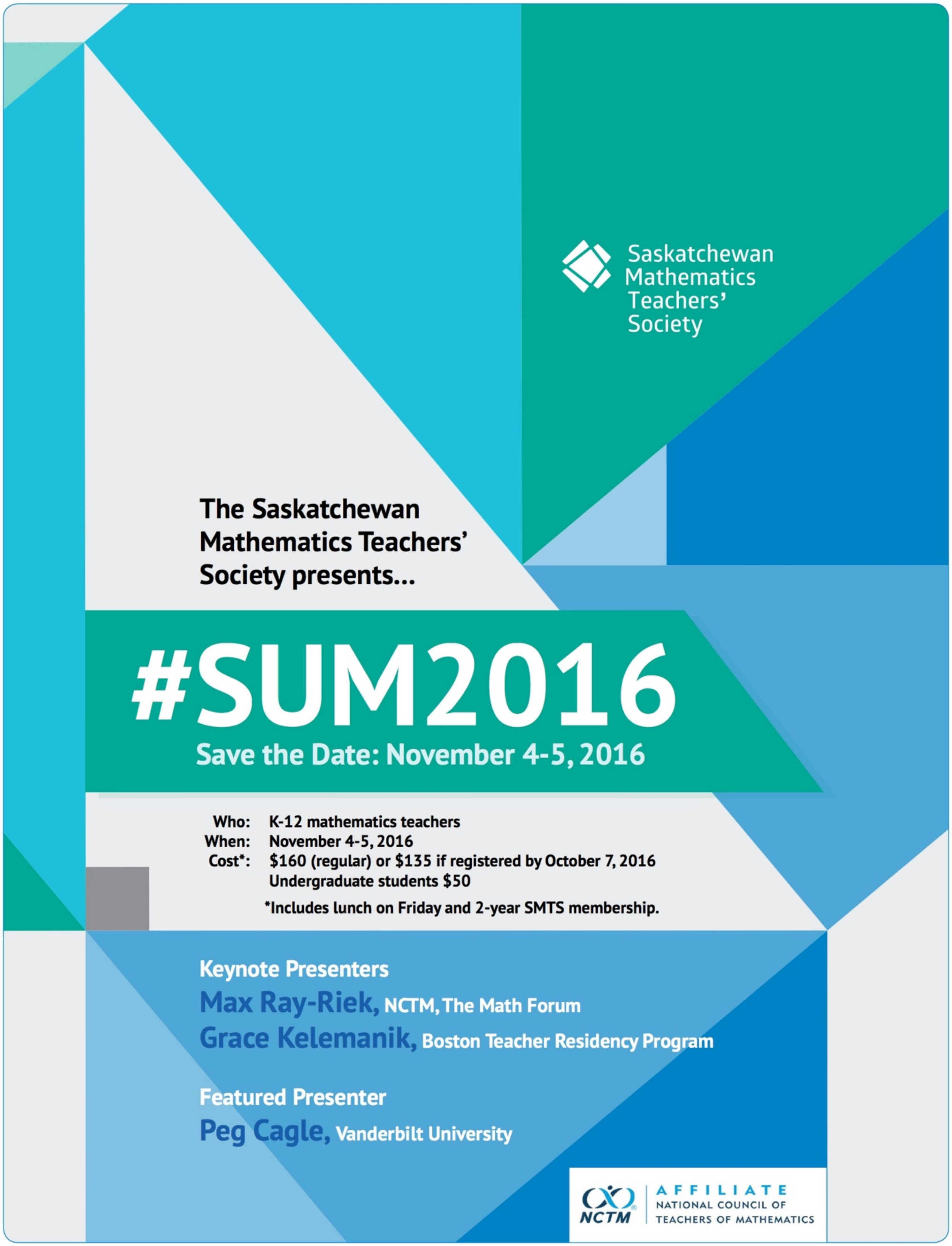
## Notice to Readers

The views expressed in each article of *The Variable* are those of its author(s), and not necessarily those of the Saskatchewan Mathematics Teachers' Society.

## Notice to Contributors

*The Variable* welcomes a variety of submissions for consideration from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants and coordinators, teacher educators, researchers, and students of all ages, although we encourage Saskatchewan teachers of mathematics as our main contributors. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Submit articles by email to thevariable@smts.ca in Microsoft Word format. Articles should be limited to 3000 words or less; authors should also include a photo and a short biographical statement of 75 words or less. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.


Saskatchewan Mathematics Teachers' Society
The Saskatchewan Mathematics Teachers' Society presents...
#SUM2016
Save the Date: November 4-5, 2016
Who: K-12 mathematics teachers
When: November 4-5, 2016
Cost*: $160 (regular) or $135 if registered by October 7, 2016
Undergraduate students $50
*Includes lunch on Friday and 2-year SMTS membership.
Keynote Presenters
Max Ray-Riek, NCTM, The Math Forum
Grace Kelemanik, Boston Teacher Residency Program
Featured Presenter
Peg Cagle, Vanderbilt University
NCTM
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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

## Message from the President

It seems only appropriate to open this month's message with a sincere and heartfelt "Congratulations!" Whether your year was wildly successful, particularly challenging, or somewhere in between, congratulations on supporting your students through an exciting year of learning during what has been a politically charged year in the educational climate of Saskatchewan.

Summer has long been a period of relaxation, rejuvenation, and reflection for teachers. I certainly hope you find time for all three this summer. While it's long been standard practice to encourage reading over the summer, I think it might be time to share some of the spotlight with some math-y fun!

Sitting around relaxing? Why not grab a bin of pattern blocks to play with and explore? They are irresistible to both young and old, and are far more social than the currently popular colouring books.

Out exploring your community? Maybe you or your family would like to join in on the Math(s) Photo Challenge! Every week, a new photo prompt with a mathematical theme is shared on Twitter (e.g., Week 1: *Symmetry*; Week 2: *Scale*; Week 3: *Lines*), after which submissions pour in from around the globe. Check out the hashtag #mathphoto16 on Twitter or https://mathphoto16.wordpress.com for more information.

Thinking you'd like to up the ante of your summer reading to include a professional development title or two? My top two summer recommends would be *Intentional Talk* by Elham Kazemi and Alison Hintz and *Powerful Problem Solving* by Max Ray-Riek. Both books are grounded in classroom examples and applications, and they have a great conversational feel. If you'd really like to geek out, I just finished *Building a Better Teacher* by Elizabeth Green, which chronicles attempts to improve both pre-service and in-service teaching in recent history, looking closely at mathematics teaching initiatives in particular. It was a fascinating look into trends in research, policy, and spending around educational initiatives.

Whatever your choice of relaxation, on behalf of the SMTS I wish you a wonderful summer. Keep an eye out for our July issue and the back-to-school edition of *The Variable*, where we'll share some of our own summer fun and some back-to-school ideas. Hopefully, as you reflect on the past school year and prepare for the one to come, you'll take a minute to jot down some of your own reflections and share them through *The Variable* – remember, all styles of contributions are welcome and encouraged!

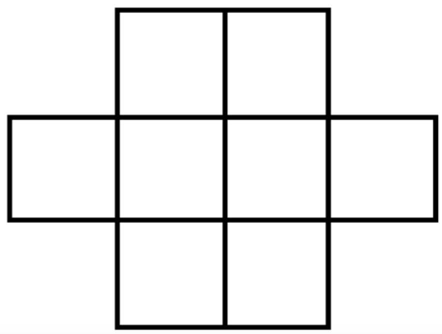
*Michelle Naidu*

## Problems to Ponder

*Welcome to this month's edition of Problems to Ponder! Pose them in your classroom as a challenge, or try them out yourself. Have an interesting student (or teacher) solution? Send it to thevariable@smts.ca for publication in a future issue of* The Variable!

**Why was 6 afraid of 7?**
*Math Challenge 2016*

Put the numbers 1 to 8 in the boxes below so that no consecutive numbers are next to each other (for example, 7 can't be directly above, below, or beside 6 or 8). Note that consecutive numbers *can* be diagonal from each other.



**Flipping coins**
*Math Challenge 2016*

There are 100 coins on a table. Each coin is numbered, and they are all arranged heads up.

First, you turn over all of the coins. Then, you turn over only the even numbered coins. Then every third coin, every fourth, every fifth, and so on. You do this until, on the very last turn, you turn over only the hundredth coin.

When you finish, which coins will be heads up? Which will be tails up?

**Patchwork**

Take square and draw a straight line right across it. Draw several more lines in any arrangement so that the lines all cross the square, and the square is divided into several regions. The task is to color the regions in such a way that adjacent regions are never colored the same. (Regions having only one point in common are not considered adjacent.) What is the fewest number of different colors you need to color *any* such arrangement?

Mason, J., Burton, L, & Stacey, K. (1985). *Thinking mathematically.* Essex, England: Prentice Hall.

*See solutions to this month's problems, courtesy of Dr. Edward Doolittle, on page 28.*

## #TDCMathFair2016

*Heidi Neufeld*

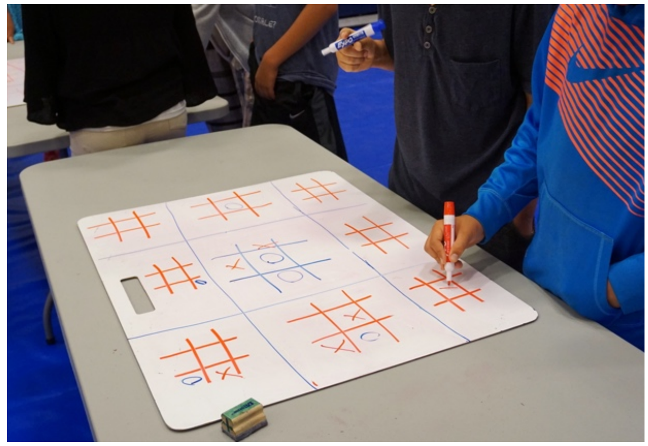


On Wednesday, June 15, Tommy Douglas Collegiate hosted its second annual Math Fair. Nat Banting, a mathematics teacher at Tommy Douglas, founded the event last year as an opportunity to stretch his Math 9 Enriched students by switching roles and allowing them to design and run math tasks for Grade 7 and 8 students. This year the fair almost quadrupled in size, welcoming over 300 students from elementary schools in the surrounding area, and prompting Banting to invite previous Math 9 enriched students to join this year's class as leaders in order to accommodate the high attendance. These students-turned-teachers-for-a-day led groups of elementary students through four stations provided to build basic skills, encourage logical reasoning, make and test predictions, and simply play with math.

At the "Sweet Sixteen" station, students used a restricted set of whole numbers to fill in a 4x4 grid of interconnected equations. At "Spa-ghet Rekt," students used spaghetti to support containers of pennies and collected the data, after which they extrapolated to determine how many pieces of spaghetti would be needed to hold a larger weight. It was a mess of predictions, pennies, broken spaghetti, and of course, a Desmos graph to collect all the data. At "Tic-Tac-Toe-Ception," students challenged each other to rounds of nested Tic-tac-toe, adjusting their strategies to account for the new dimension to the classic game.

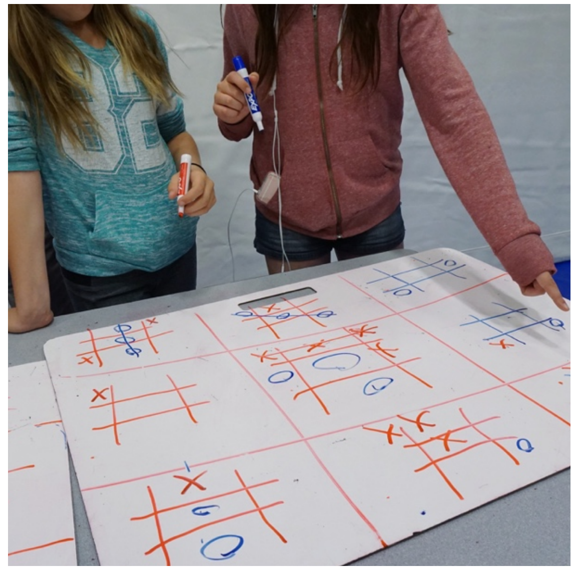
While students rotated through these stations set up in the gym, a completely different experience awaited them at the "Dark-Room Escape" station, inspired by the recent trend of entertainment-focused escape rooms in Saskatoon and beyond. Students entered the room and met a series of clues and puzzles they needed to solve in order to get out within a set amount of time. In addition to the four stations, students were encouraged to use their estimation skills by guessing the number of ball-pit balls in a 2-ft clear cube. Finally, several tables of tiling shapes (shipped all the way from Minnesota by Christopher Danielson, @trianglemancsd on Twitter) invited students to play with geometry, create patterns, and enjoy the beauty of mathematics. As one student leader said, "I should get some of these [tiles] and replace my TV. I think they're just as addicting."

As students were challenged with new ways of interacting with math, they were led by their perseverance and curiosity. Often, a student would complete a challenge, and then go find their teacher to show them their work with pride. While it was amazing to see elementary students reason mathematically at a variety of levels, what was maybe even more exciting was seeing the student leaders interact with their groups. In particular, veterans of last year's Math Fair jumped in with extra enthusiasm, getting to know the elementary students, introducing them to the various stations, providing prompts, and asking questions to help their group articulate and clarify their thinking. The influence of their own teacher was clear in the way they took on the role of educators for the day. Case in point: When asked about the solutions to a Sweet-Sixteen puzzle, one student leader stated, "There is no answer key! I don't know any more than you do," evidencing the focus on reasoning rather than only right-answering in their math teacher's classroom.



The old adage, "teaching is the best way to learn," came alive at the Math Fair as the line between teacher and student was blurred, and learners of all ages were invited to explore, reason, and play with math.

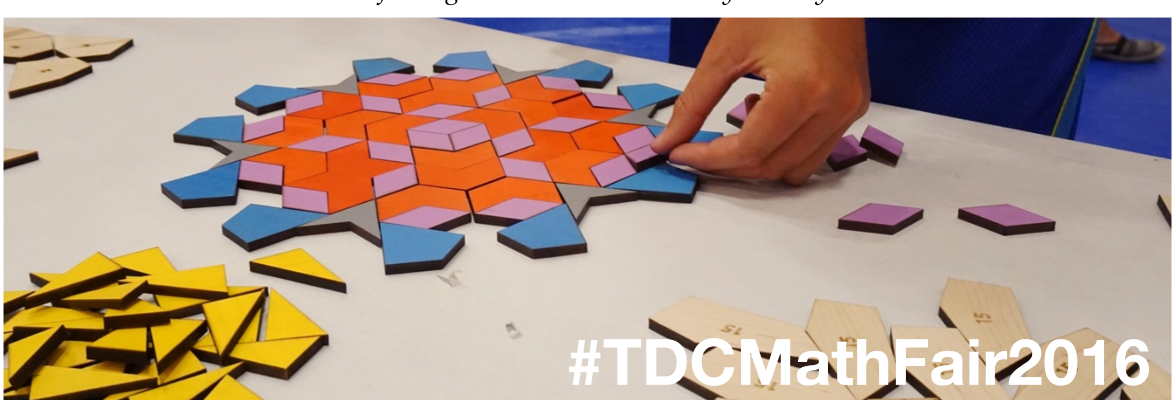
*Heidi Neufeld is a Bachelor of Education graduate from the University of Saskatchewan and a director for the SMTS. During her internship and her time at the College of Education, she has grown in her excitement to become a teacher who values deep conceptual understanding in her mathematics classroom. Heidi loves learning and collaboration, is a proud aunty, and tweets about math education at @HeidiLNeufeld.*



*Spa-ghet Rekt*

*Tic-Tac-Toe-Ception*

*Learners of all ages were drawn to the symmetry station*




*The pentagons move into the third dimension!*

*8 out of 15 pentagonal tilings completed!*

## Reflections on Math Fair 2016

*Diana Naumova, Grade 9*

I have always enjoyed mathematics; it has always been more than just numbers and solving problems to me. However, even though I've been in many different math classes and math programs, there has only been one class that was different from any other: Mr. Banting's class. He showed us how to do mathematics in a different way, in a way that no other teacher has ever done before. He basically taught us a new language. After having taken his class, I feel like a new world has been opened to me. I've never thought about mathematics as I do now. It's more than just solving some problems from a book—it's more like creating a new world, speaking a new language. Having seen math in a new way this year, I'm now even considering pursuing math in the future!

> "I've never thought about mathematics as I do now. It's more than just solving some problems from a book—it's more like creating a new world, speaking a new language."

The best part of Mr. Banting's class was probably organizing the Math Fair. I've never done anything like that before—it was new to me and very unusual. We prepared for it for two months, and it was a great time: All of us working together, trying to create new and interesting problems for Grade 6, 7 and 8 students was amazing. At first, we were still those kids that tried to do textbook questions just to forget about them the next day, but the more we were getting into math, the more our views on mathematics were changing. After some time, I finally felt like I could really understand mathematics and do it. At some moments I felt like a mathematician—and let me tell you, this feels awesome. We had a lot of arguments in our class, but that is part of being mathematical.

Closer to June, we started to seriously think about what we would need for the Math Fair and what the stations would be. We emailed the closest elementary schools. All of them responded, so we had to make sure that more than 150 people would be able to participate at the same time. At first, it was hard. We didn't know what the stations could be, we didn't know how much we'd need to buy, but we figured it out. Our four stations were: Spa-Ghet Rekt (one of my classmates came up with the name), Tic-Tac-Toe-Ception, Sweet 16, and the Dark Room. Our class was responsible for Tic-Tac-Toe-Ception, Sweet 16, and the spaghetti station, and the Foundations 20 class was working on the dark room escape.

To make sure that the stations wouldn't be too hard, we had to try all of them ourselves. We played Tic-Tac-Toe-Ception in class and it was super fun, so we thought that elementary students would like it. Then we tried Sweet 16. That one required more thinking, so we thought that in case students didn't like it, we should have a symmetry station ready for them if they wanted to take a break. The last station was the spaghetti station (Spa-Ghet Rekt). It was designed to use kids' ability to think ahead—they would try to predict how much spaghetti would be able to hold a 2.5-pound weight, knowing how much only 1 or 2 spaghetti pieces can hold. After we picked the stations, we also added an estimation station so that kids could have the chance to win a t-shirt and a game.

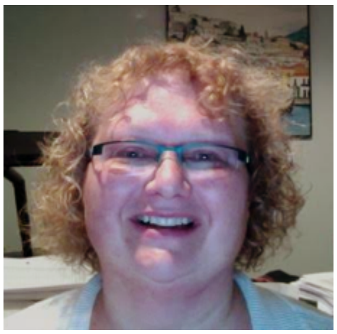
But all of that was the easy part! The harder part was creating a schedule so that each group met the same group only once in a row. My group was making a schedule for the afternoon and we didn't need to have many corrections on it, which surprised us. The next thing we had to work on was the arrangement of tables so that everyone would have an opportunity to participate. Because there were so many elementary students coming, we had to add more tables and ask some Grade 10 students to be leaders as well. We didn't really know how it would work out until Wednesday, the day of the fair.

When the day of the Math Fair came, it was truly amazing to see everything come together. We were the leaders, and we kind of had to act like teachers: We had to explain everything to the students and show them how it worked, helping them when they didn't understand. It felt good. When I saw all of those kids having fun with what we created, it made me so happy. All of that work paid off, and I was proud of what we had done. That day let me show all the beauty of mathematics to them, and it was a very special job. It was an amazing experience and I'm glad that I had the opportunity to be involved in it.

## Spotlight on the Profession

In conversation with Dr. Gale Russell

*In this monthly column, we speak with a notable member of the Western Canadian mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Gale Russell of the University of Regina.*



*Gale Russell is a Saskatchewanian through and through. She was born and grew up in Saskatoon, and after completing a B.Sc. (Honours) in Mathematics and a B.Ed. (Great Distinction) at the University of Saskatchewan, she began teaching in the community of Raymore. There, she taught all of the secondary level mathematics courses as well as some Arts Education classes. Gale was also a representative to the local teachers' association, ran a successful drama club, held regular meetings of a calculus club, and was an on call "jewelry coach," while also continuing to pursue her other passion – playing the bagpipe. During her time in Raymore, Gale also became involved in being a pilot teacher of the then-renewed high school mathematics curricula (the former Math 10, 20, A30, B30 C30), and was later regularly seconded by the Ministry of Education to be an implementation leader around the province for those curricula. After six years in Raymore, Gale moved to Rosetown, where she taught secondary mathematics while continuing her other activities at the school level and for the Ministry, and playing in a pipe band.*

*After two years and one month in Rosetown, Gale was made the first full-time permanent Educational Consultant for K-12 Mathematics at the Ministry of Education in Regina. In this role, Gale was actively involved in facilitating professional development throughout the province, in reviewing resources, in curriculum framework renewal with the Western and Northern Canadian Protocol (WNCP), and in writing the most recent mathematics curricula. Also during this time, Gale obtained her M.Ed. from the University of Regina, focusing her research on teachers' and students' conceptions about zero. After 11 years and 11 months at the Ministry, Gale left to pursue her PhD in Education at the University of Saskatchewan, focusing her research on the kinds of knowledge and ways of knowing valued within mathematics and the teaching and learning of mathematics. Just this month, Gale successfully defended her dissertation on this topic, thereby completing all of the requirements for her PhD. For the past two years (and continuing onward), Gale has been working in the Faculty of Education at the University of Regina as an Assistant Professor of Secondary Mathematics Education. She continues to play her bagpipes and has two small dogs, Euclid and Chevy.*

*First of all, thank you for taking the time to have this conversation during this busy time of the year! Could you talk a little bit about the courses are you currently (or have just finished) teaching at the University of Regina?*

Over the past two years, I have taught courses at the elementary level (EMTH 310), middle level (EMTH 217), and secondary level (EMTH 300, 351, and 450). As EMTH 310 is the only mathematics methods class that the elementary pre-service teachers (currently – note the optimism) take, it is a fast and furious class in which I try to teach the students the elementary curriculum content using the pedagogical strategies that the new curriculum (and research) supports. I also strive to have the students understand why the teaching of mathematics needs to change and to become more aware of other issues that are present in or impact mathematics classrooms (standardized testing, math anxiety, gender and cultural gaps, and so on). Actually, the description I just gave can be applied to the other classes I teach, only they are a little less intense because the students take at least two EMTH courses in the middle and secondary programs (not that we couldn't use more time in all the classes!).

Probably the biggest challenges that I give to my students in any of my classes are to contemplate what we have to directly teach to students and why, and how to engage in meaningful mathematics teaching and learning through the use of open tasks (inquiry, problem solving... pick the lingo of your liking) and class discussions. I also try to give my students experiences in the classroom or out in a school that they likely haven't had before, such as having an Elder come to our class for a day, visiting a classroom that is taught entirely through inquiry, and visiting Campus Regina Public to see an example of an alternative approach to high school and integration. I also like to engage the students in current issues in education, whether it be responding to a community paper column making claims against the teaching of mathematics or taking on the role of a BC mathematics teacher who, after 14 lost school days, has to start teaching a particular course on the following Monday. I try to make my classes as real and problematic as teaching can be, and I'm honest and open about my planning, my errors, and my quick changes in plans with my students so that they can better come to understand the process of teaching and the impacts upon it.

*I understand that your first Bachelor degree was in mathematics. What drew you to education, and then to research in the field of mathematics education?*

I actually knew I wanted to be in education when I was in high school; however, it seemed like every second person (from my graduating class of over 600) was going into education and I started to doubt whether I really wanted to be in this field. So, I started off as an English major pursuing an Arts and Science degree. For some reason, still unknown to me, I decided to change to a math major for my second year. Throughout my four years in the Arts and Science program, I paid my bills by tutoring mathematics, both to individuals and to groups. This experience reinforced what I had originally knew – I love helping people learn. From there it was a natural step to go into an education program.

My desire to do research in the field of mathematics education probably started in Grade 1, when I was drawn to helping a classmate with her math and was puzzled by why she was having so much difficulty with it. As I grew older and progressed through my K-12 schooling, Bachelor of Science, and Bachelor of Education, and started teaching in rural Saskatchewan, these kinds of experiences continued to occur. I have always been an "I want to understand why" kind of person, and I was finding that I often couldn't find satisfactory answers in the literature and research. For me, that meant that if I was going to continue teaching, I needed to start finding out the answers for myself. Of course, no answer ever stands without the support of

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questions, and I found myself hooked. To be perfectly honest, my choice of my first research topic (elementary teachers' and students' understandings of zero) was a knee-jerk reaction to being told that the concept of zero was too difficult for young children. Based upon my experiences with young children, I couldn't see how this could be true, and the notion that this concept was developed late in mathematics history was unfathomable to me. I ventured into that research to find out what was going on (and, I will admit it, to prove that this notion was false). Once I started doing this kind of research, there was no looking back, and no shortage of ideas of where to go next.

*I know that part of your research involves investigating the relationships between the teaching and learning of mathematics and culture, especially Aboriginal cultures. Although interest in this intersection is growing, I think that many still feel that mathematics—especially the Western conception of mathematics—is a "universal" language that is untainted by culture ("acultural," if you will). Could you speak a bit about the ways in which mathematics* is *influenced by culture, and perhaps also the ways in which it transcends it (if at all)?*

A few years ago, I would have said that my research is about the relationships between the teaching and learning of mathematics and culture, but now I tend to look at it more as the relationship between different kinds of knowledge and ways of knowing and the teaching and learning of mathematics. I'm still concerned with culture, and I do tend to turn to Indigenous cultures to inform and contrast with my thinking; however, I have come to view true mathematics (versus just Western or academic mathematics) as acultural. Mathematics, and how you know, understand, and work with mathematics is a consequence of the kinds of knowledge and ways of knowing that you value. Mathematics is, in fact, tainted not so much by culture, but rather by worldview. Within a specific culture, you may see tendencies or trends in the ways that people think about, do, and use mathematics, but even within that culture it is likely not universal. We have to be careful here, because when talking about worldviews, particularly when sharing a cultural name, it is easy to equate culture to worldview, and members of a culture to people holding a particular worldview, but this is far from the case. Just because I am Western does not mean that I have a Western worldview.

> "I would argue that what is universal about mathematics is that all humans consider and work within situations and solve problems that deal with mathematical notions. What is not universal is how we think about, represent, and report on these notions."

I would argue that what is universal about mathematics is that all humans consider and work within situations and solve problems that deal with mathematical notions, such as quantity, patterns, shape and space, data, and likelihood (and likely many more things I have not listed here). What is not universal is how we think about, represent, and report on these mathematical notions. A deep understanding of quantity is not restricted to those who have a number system with open and closed operations on different kinds of numbers. A deep understanding of quantity is also present in communities where they only have number words for 1, 2, and 3, and have no written representation of them. For example, for addition to be accomplished, in other words to aggregate two quantities or augment one quantity by another, does not require an operational symbol or an equal sign (as Western mathematics would have us believe). What it does require is an understanding of quantity as you represent it and an understanding of the situation that is leading you to determine the result of aggregation or augmentation (you don't even need to know those two words).

It has been clearly shown that infants under nine months of age understand that 2 – 1 = 1, they just don't know the representation "2 – 1 = 1."

Mathematical representations, algorithms, procedures, processes, and labelling, which is what many people think of when speaking of (Western) mathematics, is not universal. In fact, I would argue that it is the assumption of singularity in representing and working with (Western) mathematics that makes mathematics the much misunderstood, dreaded, feared, misused subject that it is. More importantly, I would also argue that the assumption that Western mathematics is universal has limited the ability of everyone to understand mathematics in more meaningful ways – and I mean *everyone*: both those who struggle with math and those who believe that they are math experts.

So, in its broadest interpretation, mathematics is acultural and universal; however, that mathematics does not currently exist en masse anywhere (to my knowledge) because of the limitations that have been placed upon what is accepted as mathematical based upon a particular worldview's assumptions about what mathematics looks like and how it functions in the world. Basically, mathematics, as you and I were taught it, and as we have taught it, is hegemonic and oppressive. Although the origins of this particular view of mathematics is often associated with Greece and the ancient Greek mathematicians and philosophers (which would suggest a single culture), historically it is known that much of what constitutes this mathematics (and even the mathematicians themselves) came from other parts of the world and was appropriated by a group of "mathematicians" at the time. This practice changed over time, in that the origins of new mathematics is formally recognized; however, there is much mathematics that was left behind in the accumulation of Western mathematics stemming from all cultures. It is this part of the broader field of mathematics which was never integrated in to Western mathematics that ultimately denies the universality of Western mathematics.

*Today, there is a very positive movement in Saskatchewan towards incorporating Aboriginal perspectives and ways of knowing in schools in order to make content accessible and meaningful to all students. However, I think that many mathematics teachers still feel at a loss when it comes to integrating Aboriginal content and perspectives in their classrooms. What advice would you offer such teachers, and what resources would you point them to?*

> "We need to shed our desire to place all mathematics and ways of representing mathematics in abstracted hierarchies."

As I will talk about again in the next two sections, I have come to think of the kinds of knowledge and the ways of knowing that are valued within mathematics classrooms as the key. We need to shed our desire to place all mathematics and ways of representing mathematics in abstracted hierarchies. Although it is true that we want our students to learn the rigours of Western mathematics, they need to be doing so in a context in which other ways of thinking and doing mathematics are valued, seen as contributing to mathematical knowledge, and as relating different ways of knowing mathematically. Based upon the response of a number of Elders that I had the privilege of meeting with during the curriculum renewal process, I strongly suggest that Saskatchewan teachers focus on the four K-12 Goals of Mathematics in the curriculum documents. In particular, because of editing over the course of time and curriculum release (the most recent mathematics curricula were released over 6 years: 2007 saw the release of K, 1, 4, 7; 2008 saw the release of 2, 5, 8; 2009 saw the release of 3, 6, 9; and then the Grade 10, 11, and 12 curricula were released in 2010, 2011, and 2012, respectively), go to this area in the curriculum documents for one of these grades: Grades 3, 6, 9, 10, 11, or 12. Once there, although all of the goals are important, focus in on the goal of Understanding Mathematics as a Human Endeavour. The bullets there will give you a good start to incorporating Aboriginal perspectives and ways of knowing. Often, as teachers we think of the notion of incorporating First Nations and Métis content, perspectives, and ways of knowing as being related to specific content, such as teepees, but that's not what this is about. This is not only about the kinds of thinking and responses we accept within our classrooms, but also about how we promote, relate and (non-hierarchically) value them.

> "Conversely, be very, very, very cautions of incorporating Aboriginal content. When this is done in mathematics classrooms, the content tends to become a mathematical artefact rather than a cultural one."

Conversely, be very, very, very cautious of incorporating Aboriginal content. When this is done in mathematics classrooms, the content tends to become a mathematical artefact rather than a cultural one. For example, many people in the past have used the teepee to illustrate a cone (and thereby feeling they have incorporated Aboriginal content), while a teepee is neither mathematically nor (most importantly) culturally a cone. The best way I know of to bring in Aboriginal content into the classroom is to use the Understanding Mathematics as a Human Endeavour goal as a way to create an open space in which students will choose to bring in Aboriginal (and other cultural) content once they feel it is a safe space to do so. You can also invite Elders in to speak with your students, but remember they will likely be speaking mathematically from *their* worldview, which can seem disjointed from what often happens in Western mathematics classrooms. That's not a bad thing, however, if you know to embrace the difference and celebrate it.

Gradually, some websites have also been emerging that are based on some very dedicated work related to mathematics with different Indigenous communities. Feel free to look at these for ideas, but always remember that this Aboriginal content may very well be foreign to your students. An example of a website to look at is Lisa Lunney Borden's (St. Francis Xavier University, NS) Show Me Your Math (http://showmeyourmath.ca). On this website, you will see lots of documentation about the Show Me Your Math program that she started in one Mi'kmaw school and which has now been adopted by many other schools across Nova Scotia. You will also find some sample inquiry projects there, but again, remember that these are projects related to Mi'kmaw culture and communities, so you need to make sure that you are acknowledging and situating them appropriately, or choosing to find something similar but more directly related to your students, school, and community.

*Let's talk about your own work in a bit more detail. I understand that you are in the process of, and very close to, completing your PhD in curriculum studies (how exciting!). What questions or issues are you exploring as part of this work, and what do you have planned in terms of your research program going forward?*

My main research question in my dissertation is "What kinds of knowledge and ways of knowing are valued within mathematics and the teaching and learning of mathematics?" Starting with "Jagged Worldviews Colliding" by Leroy Little Bear (found in Marie Battiste's *Reclaiming Indigenous Voice and Vision*), I complied a theoretical framework based upon two worldviews: an Indigenous Worldview and the Traditional Western Worldview. These two worldviews each define a distinct view regarding the kinds of knowledge and ways of knowing that are of value. In a very brief summary, and very Traditional Western Worldview way of just the facts, these two worldviews are characterized by the valuing of the following:

- *Traditional Western Worldview:* Linear, singular (right way, correct answer), hierarchical, abstract, compartmentalized, de-contextualized, rational, written knowledge.
- *Indigenous Worldview:* Knowledge for, of, and through relationships, contextualization, physical, emotional, spiritual, intellectual, intuitive, experiential, and cultural knowledge, diversity of knowledge and ways of knowing, connection to place, and multiple forms of representation (including oral).

In a nutshell, my work looked first at my own mathematical experiences through the two worldview lenses to take note of the kinds of knowledge and ways of knowing that I was valuing. I then considered a number of areas within mathematics and the teaching and learning of mathematics, including the philosophies of mathematics (and there are a lot of them!), the math wars, Indigenous students and mathematics, ethnomathematics, and risk education (as a consideration of possibilities for curricula development), and analyzed their alignment or misalignment with each of the worldviews.

What this research and analysis has led me to are the following conclusions:

1. Although the two are distinct worldviews, an Indigenous Worldview includes all of the ways of knowing and kinds of knowledge valued within the Traditional Western Worldview, while the reverse is not true.

2. Teaching and learning, as well as just thinking about mathematics from a grounding in an Indigenous Worldview provides a likely solution to the math wars in which both sides can survive and thrive. Likewise, this same grounding could help Indigenous students (as well as all others) in their struggles to learn and succeed in mathematics, would give value to previously unvalued mathematics, and would provide students, teachers, and society with broader and richer ways of thinking and knowing mathematically. I called such a grounding the *Transreform Approach to the Teaching and Learning of Mathematics*, as it acknowledges and values the traditional and reform approaches, the so-called middle ground between the two, and approaches to mathematics that so far may not have even been considered.

> "Teaching and learning, as well as just thinking about mathematics from a grounding in an Indigenous Worldview provides a likely solution to the math wars in which both sides can survive and thrive."

3. To engage in the Transreform Approach, we also need to re-view and re-new our philosophies of mathematics.

So, where to go now? I have done some preliminary work with a group of teachers that ultimately provided rich data about how easily values from the Traditional Western Worldview can slip in and take over (I call it "intrusions") how one is teaching, without the teacher ever realizing it. This same study also identified how group discussion and reflections can help to thwart such overtaking.

From here, I want to take a step back and consider the question of "What ways of knowing and kinds of knowledge are being valued in mathematics classrooms in Saskatchewan?" Then, I hope to build on this information through an exploration of how to effectively help teachers transition in their worldview groundings and to analyze the impacts (good and bad) upon student achievement and affective responses.

As is typical of the person I am, I also have a number of side research interests. Ultimately, I believe that there will be ties into my worldview research, but I am also really interested in the questions of "Why do pre-service teachers pick a particular level (elementary, middle level, secondary) to specialize in, and why are they not picking a different one," "What is 'common sense' when it comes to the teaching and learning of mathematics," "What instances of polysemy (multiple meanings for a single word) are occurring in mathematics classrooms in communities where Aboriginal English is spoken," and "How to engage communities and parents in understanding changes in the teaching and learning of mathematics (and to help support them in supporting their children with mathematics)." Of course, there are many others that are floating around my mind, but it gives you a bit of a sense of the diversity of where my interests lie.

*Your work has been published in a variety of journals, books, and conference proceedings, including the* Canadian Journal of Science, Mathematics and Technology Education *and our very own* vinculum *(the journal of the Saskatchewan Mathematics Teachers' Society). Which of your publications would you recommend to mathematics teachers in the province who are looking to grow in their practice and their understanding of the teaching and learning of mathematics?*

So this is a tough question to answer, because it depends upon what people are interested in. I would first, however, recommend a publication which is not mine, but that I mentioned previously – that of Leroy Little Bear:

> Little Bear, L. (2000). Jagged worldviews colliding. In M. Battiste (Ed.), *Reclaiming Indigenous voice and vision* (pp. 77-85). UBC Press: Vancouver, BC.

For me, this was my big eye-opener in coming to understand what people were talking about when they said things like "First Nations and Métis have different ways of knowing." Although it's not explicitly talking about mathematics, I found myself writing in the margins comments such as "this is how the teaching and learning of mathematics has been done" and "this is how research says it should be done."

If people are interested in a discussion of the math wars and the struggles of Indigenous students in light of the two worldviews, then I recommend "The Marginalisation of Indigenous Students Within School Mathematics and the Math Wars: Seeking Resolutions Within Ethical Spaces" by Egan Chernoff and myself. This article can be found in Volume 25, Issue 1, p. 109- 127 of the *Mathematics Education Research Journal*. If, instead, people are interested in how changing the kinds of knowledge and ways of knowing that are valued in mathematics classrooms could impact curriculum and lives, then I suggest reading "Risk Education: A Worldview Analysis of What is Present and Could Be" which can be found in The Mathematics Enthusiast, Volume 12, Issues 1-3.

*Thank you, Dr. Russell, for taking the time to share your research and perspectives with our readers. We'll be following your future work with interest!*

*Ilona Vashchyshyn*

## Negative zero

*Cindy Smith*

"The answer is negative zero."

–Kenny N.

As part of a "connected educators" Twitter/blog exploration I am part of, I am asked to answer one of two questions in a blog post. This month, I had the choice to write about an open-ended activity that I use, or about something that made my classroom unique, made it my own. For the last couple of days, I had been thinking about what I might write; I love open-ended activities and have many good examples to draw from. I was prepared to embark on a rich description of my favourite collaborative graphing activity, when something happened. What happened? Kenny N. happened.

Running some quick errands between school visits, I pull up to a stop sign. Beside me a red truck pulls up, stereo blazing, girlfriend (wife?) in passenger seat, cargo full of furniture. I look over, and the driver is rolling down his window. The face looks familiar… I know I taught this guy… but after teaching up to 120 kids a semester for 11 years, well… sometimes, I can't remember their names.

Me: "Hey! How you doing?" Him: "Awesome, how 'bout you?" Me: "Groovy. What's with all the furniture? You movin'? Or opening a furniture store?" Him: "Yah, and we needed a new bed, and…" Blah, blah, blah, the conversation continues. Both smiling, teasing, laughing. The light turns green, but we are still driving side by side and yelling back and forth. I hated to ask, but I needed to know: "Hey, remind me of your name again…?" Him: "It's me! Negative Zero! Kenny N.!" (In

> "Negative Zero. I had taught this kid for three semesters in a row. Math was not his 'thing'—well, school and conformity weren't his 'thing' in general…"

my defense, the guy had lost a ton of weight and looked *way* different… but it all came flooding back.)

Negative Zero. I had taught this kid for three semesters in a row. Math was not his "thing"—well, school and conformity weren't his "thing" in general… but he had a big heart, and he was just funny.

I used to post quotes by famous mathematicians around my room in an effort to immerse my students in a culture of mathematics. "We seek for truth, not only through reason, but also by the heart" (Pascal). And we'd talk about this quote. Was Pascal speaking as a philosopher? Or as a mathematician? Then, when we were doing geometric proofs: "God eternally geometrizes," "Let no one ignorant in geometry enter here" (Plato), and "Proof is the idol before which the pure mathematician tortures himself" (Eddington). Others: "This was done very elegantly by Minkowski, but chalk is cheaper than grey matter and we shall take it as it comes"—Einstein, referring to what he viewed as his own weakness in mathematics. We would discuss these and other quotes. What was the author saying? What does it reveal about this crazy, historical, fanatically-pursued discipline we call mathematics?

> "He was so struck to see his words done up in fancy font, printed, and put up on the wall"

But the fun part was when my students would say something profound, and when I would quote *them*. I would type up their quote and, without saying anything, put it up among the others, waiting for someone to notice. And so it was, a few days after a particularly taxing debate about a certain problem and some interestingly misguided solutions proposed by Mr. N., I put up the quote: "The answer is negative zero" –Kenny N. I remember that he was so struck to see his words done up in fancy font, printed, and put up on the wall. And so, many, many years after he graduated, we both still remembered the debate and his crazy theory involving negative zero. And, as I said then, who knows? Maybe he was proposing something someone will build on some day (if you are some calculus nut, please don't email me some lecture on the virtues of this in terms of limits!).

Another student quote that made the list: After a lively debate about whether the vertices on a kite are congruent, a riled-up student, remembering some great hands-on activity by a teacher who taught him during the middle years (thank you, whoever you are!), finally shushes his arguing group by holding up a kite cut out of paper. He points to it and says, "You can't fold them corners, boys!"

There was also the mathematically gifted student who used graphing calculators to construct stars with 6, 7, 8, 9, 10, and even 11 vertices, and then inscribed ellipses (made of two parabolas). He gathered a little cult-like following of kids who wanted to abandon the graphing quadratics lesson to make these great stars like him (which really took a lot of thinking and many equations!). Overheard between this student and one of his protégés: "Even-pointed stars are for the weak. Go for odd points!" (I added the quote to my wall: "Even-pointed stars are for the weak" –Tyler B.)

At the time I was just being goofy, but the kids would come back to my room after they were done my class, just to see if their quote was still up. And there was much talk and excitement if someone made the "wall." I always encouraged them by telling them stories about mathematical discoveries made by unexpected people, sometimes young people.

Like the little kid who named the googol, the sixth graders who created a triangle much like Pascal's, the high school drop-out who proved a university professor wrong about tessellating hexagons, and so on. I told them that Einstein felt that he wasn't a mathematician (!) and that he said that once the mathematicians got hold of his theorems, even he didn't understand them anymore. If we thought that someone in our class discovered something novel, we would look it up, and would sometimes even submit it to a university. (Once, we thought we came close to a discovery with a sequence; as it turns out, we didn't discover a new sequence, but I believe we did find a novel application. I'll tell you about it sometime.) Anyway, I often pointed out that great discoveries are made by people that ask interesting questions, and that, as Socrates said, "Wonder is the beginning of wisdom". I made them feel that mathematics was a vibrant, living field that was still in development, and that they could contribute to it. And they remembered that.

"I made them feel that mathematics was a vibrant, living field that was still in development, and that they could contribute to it. And they remembered that."

So here's to Kenny N., Mr. Negative Zero. Thanks for reminding me of this!

P.S. The "Twitter exploration" I'm involved in (when I have time!) is called MTBoS—the "Math Twitter Blogosphere." Math teachers from all over the world come together to share their blogs and ideas. It's a great source of inspiration—I've picked up many cool classroom ideas from there. Check it out! Follow the hashtag #MTBoS.

*This article is adapted from a post originally published on October 8, 2013 on Cindy's blog, http://blogs.gssd.ca/csmith/*

*Cindy Smith taught high school math at Yorkton Regional High School for 11 years, and in 2008, she was honoured with a Master Teacher award from the Saskatchewan Mathematics Teachers' Society. After completing a Stirling MacDowell research project on Technology-Supported Inquiry in Senior Math and then completing a Master's degree, Cindy took up her current position as Senior Mathematics Coach for Good Spirit School Division. Her focus is on innovative teaching, inquiry, technology, and classroom culture.*

# Encouraging mathematical habits of mind: Puzzles and games for the classroom

## Domino puzzles and rectangles

*Susan Milner*
*Professor Emerita, Department of Mathematics & Statistics, University of the Fraser Valley, British Columbia*
www.susansmathgames.ca

One of my great pleasures in the past five years has been visiting K-12 classrooms, sharing puzzles and games based on mathematics and logic. It is very satisfying to watch students, including some of the most reluctant or math-averse, get drawn into the activities, have fun, and learn way more than they realise. It's delightful to hear comments such as:

> "Hey! I can *do* this!"
> "I'm ready for the next harder one."
> "Do we have to stop now?"
> "Can we take this home to work on?"
> "Best math class ever!"

I've been invited into over 300 classes and shared the pleasure of hundreds of teachers as their students revealed logic and problem-solving abilities they might not have shown during regular class time. And I've shared with many teachers that wonderful feeling when the entire classroom falls silent as the students concentrate hard.

As math educators, we are all aware of the huge difference that motivation makes for students. If all that playing with puzzles accomplished were to pique students' curiousity, create some excitement, and provide incentive to finish their "regular" work, then puzzles would be a Good Thing. But playing with puzzles provides far more: students develop mathematical habits of mind that will stand them in good stead for many aspects of their future. They notice patterns and ask whether those patterns are accidental or essential; they question their assumptions; they further their ability to think logically and to express their thinking clearly; they develop their abilities to reason about spatial relationships, both two- and three-dimensional. They persevere, starting over when necessary, and they develop confidence in approaching new situations. It is remarkable how much harder many of us will work if what we are doing is fun, not work!

> "If all that playing with puzzles accomplished were to pique students' curiousity, create some excitement, and provide incentive to finish their "regular" work, then puzzles would be a Good Thing. But playing with puzzles provides far more."

Math/logic puzzles provide a model for the fundamentals of problem-solving:

- Make sure you understand the question/goal/rules
- Look for things that you know *for sure*, not just what might be true
- Be able to justify each step that you take so that it makes sense to someone else

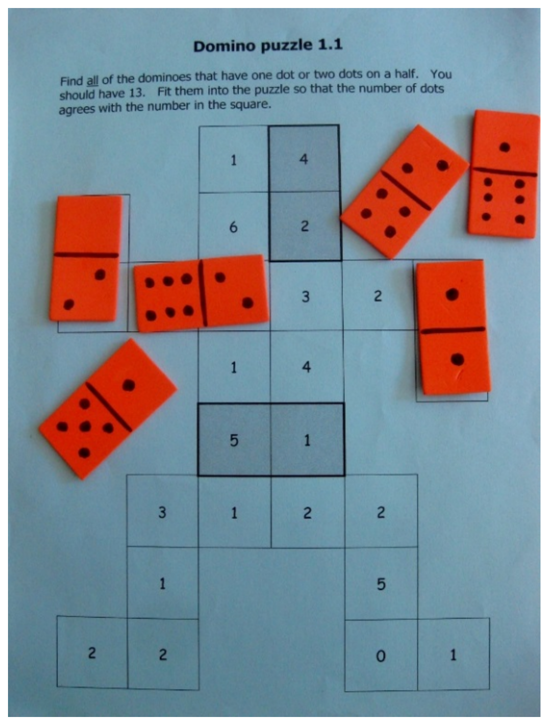
- Work on being aware of any choice you do make, so that if you end up with a contradiction, you can return to that choice and correct it (not the same as guessing!)
- Check that what your final result is correct – as no answer is provided with these, you have to look for any errors yourself

I'm greatly honoured to have been asked to write for *The Variable*. The plan is to present a couple of puzzles at a time, one that has been successful with younger students and one that has been successful with older students. I am starting with two types that are easy to get started on, good for the beginning of the year.

Templates for all of the puzzles I'll describe are available at www.susansmathgames.ca, along with descriptions of methods I've found effective for introducing puzzles to a class.

**Domino Puzzles**

*The goal is to match up the numbers on a set of dominoes to the numbers on the puzzle. Students in all grades can find something to challenge them, at one level or another.*




There are four levels on my website, each more difficult than the previous level. Within a level the puzzles get gradually harder – at least, that is the goal! Feedback on the level 2-4 puzzles has been not entirely consistent.

If your class gets hooked, they'll want to try harder and harder puzzles. Students who get really hooked might enjoy creating their own puzzles for each other.

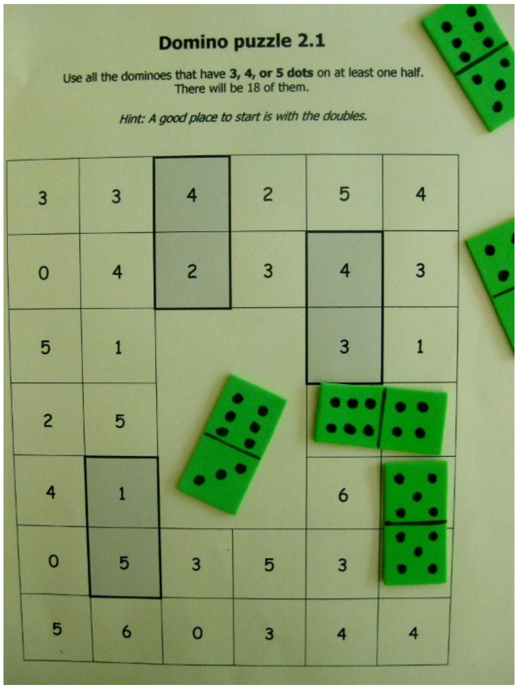
**Level 1 puzzles** use only 13 dominoes and are excellent for Grades 1-3, right from the beginning of the year. Kindergarten students enjoy them a bit later in the year. Grade 4 students find the harder Level 1 puzzles engaging in the beginning of the year, but will quickly move on to the next level.

A hint many youngsters find helpful: Start with places where we have no choice, such as the shaded pairs and the bits that stick out.

For the youngest children, these are also good for practicing *subitizing*, a word I learned this year from a kindergarten teacher (it means recognising the number of dots without having to count them).

**Level 2** puzzles use 18 dominoes and are engaging for Grades 4-7.

Level 1 and 2 puzzles use dominoes that are 1 inch by 2 inches, standard size. If you have class sets of dominoes, they should work, but I prefer making them out of craft foam in




different colours, so that it's easier to keep track of individual sets. The templates are on my site, and the restriction is stated at the top of each puzzle.

If you have time, it is a good exercise to start each student with a full set of dominoes and have them follow the given restriction to sort out the pieces for the level they are working on.

**Level 3 & 4** puzzles use all 28 dominoes, and can be quite challenging! I've used them with Grades 8-12 at summer math camps and as part of the activities associated with math contests – that is, with self-selected students who already enjoy math. I have not tried them with an ordinary class and would love to hear about it if you do.

See my website for templates for the puzzles and for the subsets of dominoes for Level 1 and Level 2.

**Rectangles**

*I've introduced Rectangles successfully to Grades 4-12 at any time of the year, and to Grade 3 students near the end of the school year. I've even tried the introductory level with a well-focused Grade 2 group!*

This Japanese puzzle is also known by its original name of Shikaku ("divide by box").

Mathematics educators typically start by giving the rules or goals for a game or problem, but I have found it far more engaging to start with a solved puzzle and have my audience figure out what the rules must have been. I show one of these two examples at the beginning of a session. Can you figure out the rules?

**Introductory level** *(younger grades)*

| | | | | |
|---|---|---|---|---|
| | | | 3 | |
| | | 4 | | 2 |
| 3 | 5 | | 4 | |
| | | | | |
| 2 | | | 2 | |

**Standard level** *(older grades)*

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 3 | | | | 2 | |
| | | | 4 | | | |
| | | | 3 | | | |
| | | 4 | | 2 | 3 | 4 |
| 3 | 5 | | 4 | | 3 | 3 |
| | | | | | | |
| 2 | | | 2 | | | |

(If I'm using a chalk- or white-board, I don't colour in the rectangles but rather outline them in different colours.) I don't explain anything but instead ask the students for their observations. It doesn't take long for them to figure out the rules, which are:

- *there is a single number in each rectangle*
- *the number tells us how many of the small squares are in the rectangle*
- *there are no gaps and no overlaps*

I find it worth confirming that squares are indeed rectangles, just as Grade 5 boys are human beings!

One reason I like this puzzle is that makes physical the difference between prime and composite numbers. It connects number theory to geometry, in that prime numbers can be drawn only one way (long and skinny) while composite numbers can be drawn in at least two ways.

Another reason I find this puzzle good in the classroom is that most students expect math problems to have only one correct answer (usually true about the problems we create for them to solve), and worse, they also often seem to conclude that there is only one correct way to arrive at that answer, or perhaps one "best" way. Solving Rectangles and other puzzles with input from the whole class at once can be used to demonstrate that there are often several equally good pathways to the answer.

Here is a worked example, demonstrating some of the logic one can use to solve a Rectangles puzzle. If you like, cover up my solution to try the puzzle first on your own, then compare your logic to mine.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | |
| | | | | | | | 4 | |
| | | | | | 12 | | 5 | |
| 3 | | | 2 | | 2 | | | 2 |
| 2 | | | 2 | | | | | |
| | 10 | | | 4 | | 2 | | |
| | 2 | | | 8 | | 2 | | 3 |
| | 2 | | | | | | 2 | 2 |
| | | 3 | | | | 4 | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | | | ** | | | | | * |
| | | | | | | | 4 | |
| | | | | | 12 | | 5 | |
| 3 | | | 2 | | 2 | | | 2 |
| 2 | | | 2 | | | | | |
| | 10 | | | 4 | | 2 | | |
| | 2 | | | 8 | | 2 | | 3 |
| | 2 | | | | | | 2 | 2 |
| | | 3 | | | | 4 | | |

- That 5 can go in only one possible direction.
- The 10 is similarly constrained, as is the 2, bottom right.
- Which number(s) can reach the the * square? *Only* the 4.
- Which number can reach the ** square? *Only* the 12.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | |
| | | | | | | | 4 | |
| | | | | | 12 | | 5 | |
| 3 | | | 2 | | 2 | | | 2 |
| 2 | | | 2 | | | | | |
| | 10 | | | 4 | | 2 | | |
| * | 2 | | | 8 | | 2 | | 3 |
| * | 2 | | | | | | 2 | 2 |
| * | | 3 | | | | 4 | | |

- Now put in a few more rectangles that have no choice.
- Again, * marks squares that can be reached by only one number. This is a very useful technique that you may need to point out several times.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | |
| | | | | | | | 4 | |
| | | | | | 12 | | 5 | |
| 3 | | | 2 | | 2 | | | 2 |
| 2 | | | 2 | | | | | |
| | 10 | | | 4 | | 2 | | |
| | 2 | | | 8 | | 2 | | 3 |
| | 2 | | | | | | 2 | 2 |
| | | 3 | | | | 4 | | |

And finish it off!

**A few suggestions**

- Try to encourage your students to scan a puzzle before they start, looking for numbers that have only one possible rectangle associated with them.
- Some people like to start with the largest number in a puzzle – that works sometimes, but not always.
- Argument by contradiction can be a very helpful tool. You can discard a possibility if it would block another number's rectangle and there would be no further choice for that second rectangle. This type of logic is used a lot in mathematical argument: "If this were true, then we'd end up with an impossible situation, so it can't be true." Testing a hypothesis isn't the same thing as guessing. If you keep the chain of argument fairly short so you know where it started, this can work well.
- Some students almost immediately want to try the hardest puzzle they can see. This tends to lead to frustration, "I hate math," and even the occasional meltdown. I tell them that they wouldn't start out running a marathon, but would instead increase their stamina bit by bit, working towards the marathon.
- Some students can "see" solutions for smaller puzzles and resist trying to articulate their logic when asked. This can create difficulties for them when they hit puzzles large enough that they can't see the answer right away. If they are really quick with the easier puzzles, I'll suggest they skip a couple in the sequence, but not too many, and I'll try to encourage them to tell me some of their logic.

Here are two puzzles for you to try. You'll be able to tell that you have done them correctly just by looking at your completed version. No need for an "answer at the back of the book" …

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 3 | | | 2 | 2 | | 3 | |
| 5 | | | | | | 3 | | |
| | | | | | | | | 8 |
| 5 | 4 | | 4 | 9 | | | 2 | |
| | | | | | | | | |
| | | | | | | | 2 | 5 |
| | 4 | | | | | | | |
| | | 8 | | 3 | 3 | | | |
| | | | | | | | 6 | |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 7 | 2 | | 2 | 2 |
| | | 2 | | | | | | 4 | | |
| | | | | 2 | | | | | | |
| | | 4 | 4 | 2 | | | | | | 7 |
| | | | | | 3 | | | | | |
| 7 | | | | 2 | | | | | 16 | |
| | 7 | | | | | | | | 2 | |
| | | | 2 | | | | | 10 | | |
| | 2 | 3 | 2 | | | | | | | |
| | | | 10 | | | | | | 15 | 2 |
| | | | | | | | | | | |

The larger the Rectangles puzzle is, the more it will test your factoring ability, as well as your logic.

Students often ask, especially if they get stuck, whether the box can go around a corner. Wishful thinking, of course, but there is a puzzle that allows us to do just that. It's called Filomino, and is considerably more difficult than Rectangles. It's a good one to encourage students to try once they can handle the larger Rectangles puzzles.

See my website for my material on Rectangles in the classroom, as well as links to on-line sources of more puzzles.

So there we are… I'd really like to know how things go if you do try one of these games in your classroom! And I'd love to hear from you about your own favourite math/logic puzzles and games. You can contact me at susan.milner@ufv.ca.

*Susan Milner taught post-secondary mathematics in British Columbia for 29 years. For 11 years, she organised the University of the Fraser Valley's secondary math contest – her favourite part was coming up with post-contest activities for the participants. In 2009 she started Math Mania evenings for local youngsters, parents, and teachers. This was so much fun that she devoted her sabbatical year to adapting math/logic puzzles and taking them into K-12 classrooms. Now retired and living in Nelson, BC, she is still busy travelling to classrooms and giving professional development workshops. In 2014 she was awarded the Pacific Institute for the Mathematical Sciences (PIMS) Education Prize.*

## Problems to Ponder, June edition: Solutions

*Edward Doolittle*

"I'm giving solutions to the June edition problems before solutions to the May edition problems, because some of the June edition results are used to solve May edition problems." [*Look for Dr. Doolittle's solutions to the May edition problems in next month's edition of The Variable! –Ed.]*

To solve difficult problems, it helps to have a strategy. This month, we can benefit from a strategy from the great teacher of problem solving, George Polya, who wrote, "The more ambitious plan may have more chances of success" (*How to Solve It*, p. 121). This is known as the inventor's paradox: sometimes, a more ambitious plan, such as solving a more constrained version of a problem, turns out to be easier.

For this month's problems, I will show how the inventor's paradox—that is, creating a more ambitious plan—, may aid in the solution of all of the problems.

**Why was 6 afraid of 7?**

In this problem, we are asked to put the numbers 1 to 8 into the boxes so that no two numbers are next to each other. We are told that consecutive numbers can be diagonal from each other.

The difficulty solving this problem is that there does not seem to be a good way to navigate the vast sea of possibilities in order to find the many islands which represent correct numbers. We may be left with a strategy of guessing, and then modifying our guess, which may in the end be effective, but is certainly not the most enlightening approach.

What is a more ambitious strategy in this case? We could add in the extra restriction that consecutive numbers cannot be diagonal from one another. The extra restriction would seem to make the problem harder, and sometimes extra restrictions do make the problem harder or even impossible. But we have nothing to lose by giving it a try, and we may have a lot to gain.

Let's think about the middle squares in the figure. If I put a middle number (4, say) into a middle square, what happens?

| | | | |
|---|---|---|---|
| | X | X | |
| X | 4 | X | O |
| | X | X | |

Then the number 3, which is an immediate neighbour of 4, cannot go into any of the boxes with an X. We see that the only place for 3 to go is the square marked with an O. But 4 also has another neighbour, namely 5, and the same reasoning applies to 5, so 5 must also go into the square marked with an O. But that's impossible; the space marked with an O cannot hold two different numbers, so 4 cannot go into a middle square.

The same reasoning applies to any number with two neighbours: in particular, 2, 3, 4, 5, 6, and 7 cannot go into the middle squares in the figure. So the only numbers that can be placed into these spots are the numbers with only one neighbour, namely 1 and 8. So a solution, if there is one, has to look like one of these:

| | | | |
|---|---|---|---|
| | | | |
| | 1 | 8 | |
| | | | |

| | | | |
|---|---|---|---|
| | | | |
| | 8 | 1 | |
| | | | |

Notice that the two arrangements are really the same, up to mirror image. So we only have to keep one, we can ignore the other, and then take mirror images when we've found an answer in the end.

Now let's think about filling in other numbers. We know that there's only one place for the neighbour of 1 to go, and there's only one place for the neighbour 7 of 8 to go, so we put them in:

| | | | |
|---|---|---|---|
| | | | |
| 7 | 1 | 8 | 2 |
| | | | |

Now we're making progress! The number 3 can only go in one of two places, so it isn't next to 2:

| | | | |
|---|---|---|---|
| | 3 | | |
| 7 | 1 | 8 | 2 |
| | | | |

| | | | |
|---|---|---|---|
| | | | |
| 7 | 1 | 8 | 2 |
| | 3 | | |

Again, the two arrangements are mirror images, so we only need to keep one and we can ignore the other for the time being. We'll keep the first. Now we fit the number 6 into the grid, which can also go in only one of two places so it is not next to 7:

| | | | |
|---|---|---|---|
| | 3 | | |
| 7 | 1 | 8 | 2 |
| | | 6 | |

| | | | |
|---|---|---|---|
| | 3 | 6 | |
| 7 | 1 | 8 | 2 |
| | | | |

Note the second possibility doesn't work out well when we have to put 4 and 5 in: it forces 4 and 5 to be next to each other. So we eliminate the second possibility and continue with the first.

The number 4 can't be next to 3, so we know where it has to go:

| | | | |
|---|---|---|---|
| | 3 | | |
| 7 | 1 | 8 | 2 |
| | 4 | 6 | |

Finally, we know where the number 5 has to go:

| | | | |
|---|---|---|---|
| | 3 | 5 | |
| 7 | 1 | 8 | 2 |
| | 4 | 6 | |

At this point, we should perform a check to make sure that the answer really does satisfy all the conditions we set. Following the sequence from 1 to 8, we can see that in no case is a number next to an adjacent number, on the side, up or down, or diagonally.

Now taking mirror images, we can see that there are exactly four solutions to this harder problem, obtained by taking mirror images or double mirror images of the answer above:

| | | | |
|---|---|---|---|
| | 3 | 5 | |
| 7 | 1 | 8 | 2 |
| | 4 | 6 | |

| | | | |
|---|---|---|---|
| | 4 | 6 | |
| 7 | 1 | 8 | 2 |
| | 3 | 5 | |

| | | | |
|---|---|---|---|
| | 6 | 4 | |
| 2 | 8 | 1 | 7 |
| | 5 | 3 | |

| | | | |
|---|---|---|---|
| | 5 | 3 | |
| 2 | 8 | 1 | 7 |
| | 6 | 4 | |

So we have not only solved the more ambitious problem we set for ourselves, but we have found all the solutions! Those four solutions are also some of the solutions to the original, weaker problem, which was our initial objective. There may be many other solutions to the original problem, but we just had to come up with one, and that we have done.

Why is it that the more ambitious problem turned out to be easier? I think the reason in this case is that for the more ambitious problem, we can construct a narrative, a story, about which number must go where at each stage. Our options are limited, so we must choose the correct option (if there is one). For the original problem, it would be much harder to construct a linear narrative. At each stage there are so many possibilities, our story line grows into a tree and becomes difficult to manage, and a wrong choice at one point can have repercussions several steps into the future, instead of immediately.

The strategy of trying to solve a more ambitious problem doesn't always work, but when it does, it can be very effective.

Here's another more difficult problem for you to try, now that you know the method. Take the 16 face cards (with values J, Q, K, and A) from a deck of cards and arrange them in a 4x4 square so that in each row, all four faces and all four suits are represented, and in each column, all four faces and all four suits are represented. The diagonals don't have to contain all four faces and all four suits.

**Flipping coins**

The first thing that occurs to me when I look at this problem is that there may not be anything special about the number 100, so I want to make it easier by reducing the size of the number 100. I don't know where I'd find 100 coins anyway, so let's try a more manageable number like 20, or even just the number of coins in my pocket (which, sadly, is rather small at the moment). That's an example of just plain making the problem easier to work with.

When we run our test, we find that the coins numbered 1, 4, 9, and 16 are tails up, and all the other coins are heads up. If you're still unsure of the pattern, try it again with more than 20 coins, say 25 or 30. You'll notice then that 1, 4, 9, 16, and 25 are tails up, and all the rest are heads up. If you know your number sequences, you'll eventually notice that all the perfect squares are tails up, and all the non-perfect squares are heads up.

Now we have a conjecture, but how can we be sure that the pattern holds when we work with 100 coins, or any number of coins? We need to apply some reasoning at this stage so we can be sure our answer is correct.

This is the point where we make the problem more ambitious. Instead of asking whether a particular coin was flipped an odd number of times (like 1, 4, 9, 16, etc., were) or an even number of times, we might ask if we can tell *exactly when* and *exactly how many* times a coin would be flipped, which seems like a more ambitious problem. Let's pick a number and try to figure out exactly how many times the coin at that number gets flipped. The coin at 12, say, gets flipped on round 1, round 2, round 3, round 4, round 6, and round 12, for a total of 6 times. On the other hand, the coin at 9 gets flipped on round 1, round 3, and round 9.

If you are good at your number facts, you'll notice a pattern: the coins get flipped on rounds which are divisors of the coin position. For example, coins at prime number positions (2, 3, 5, 7, 11, etc.) only get flipped twice, because the position number only has two divisors.

Now we have a better handle on the problem. At this point, we should be asking ourselves whether there's a reason why square numbers have an odd number of divisors, while non-square numbers have an even number of divisors. Let's write out all of the ways that we can obtain 12 by multiplying two numbers, and all of the ways we can obtain 9:

12 = 1 x 12
12 = 2 x 6
12 = 3 x 4
12 = 4 x 3
12 = 6 x 2
12 = 12 x 1

9 = 1 x 9
9 = 3 x 3
9 = 9 x 1

There's a kind of symmetry to these arrangements. We can rotate them 180° around a center point, and the arrangements remain unchanged. Now we can see what is special about the square numbers like 9: there's a row in the exact middle of the arrangement (e.g., 9 = 3 x 3, or for the arrangement for 25 we have 25 = 5 x 5), so squares have an odd number of factors.

On the other hand, non-squares do not have a row in the exact middle of the arrangement, so they have an even number of factors. (Lines in the top half of the arrangement are paired with lines in the bottom half.)

Now we can see clearly why coins at square number positions end up tails, while coins at non-square number positions end up heads.

Here's a similar, but more difficult problem to try. First, we need to know what a *squareful* number is (reference: [https://oeis.org/wiki/Squareful_numbers](https://oeis.org/wiki/Squareful_numbers)): A number is called squareful if it is divisible by a perfect square bigger than 1. The list of the first few squareful numbers is 4, 8, 9, 12, 16, 18, 20, 24, 25, 27, 28, …. Notice that the list contains every perfect square, but it includes many other numbers as well, such as every multiple of 4, every multiple of 9, and so on. On the other hand, the set of squareful numbers does not include primes, for example.

We start again with 100 coins, all heads, and we proceed in a manner similar to before. On our first round, we flip every coin at a multiple of the number 1, so coins at positions 1, 2, 3, 4, …. On our second round, we flip every coin at a multiple of the number 2, so coins at positions 2, 4, 6, 8, …. On our third round, we flip every coin at a multiple of the number 3, so coins at positions 3, 6, 9, 12, …. However, on our fourth round, we don't do anything because 4 is squareful. On the fifth round, we flip every coin at a multiple of the number 5, so coins at positions 5, 10, 15, 20, …. In other words, we proceed just as we did in the original problem, except that if a number is squareful, we move immediately to the next round without flipping any coins. At the end of this process, which coins are heads up, and which are tails up?

**Patchwork**

For this problem, we can be really ambitious: we can ask whether it's possible to colour the regions with one colour. If you draw a picture of a square with one line dividing it into two parts, you'll see that one colour doesn't work. Even though it's a trivial observation, it's progress, because now we know that the minimum number of colours is greater than 1.

Let's continue being ambitious. Let's see whether we can colour the regions with the next smallest number of colours, two colours. In the previous picture, we could, obviously, because there are just two regions. But now let's add another line. We can make 3 regions with two lines, or we can make 4 regions. Clearly we can colour 3 regions with 3 colours and 4 regions with 4 colours, but that's not being ambitious. Let's try to see whether we can get away with 2 colours.

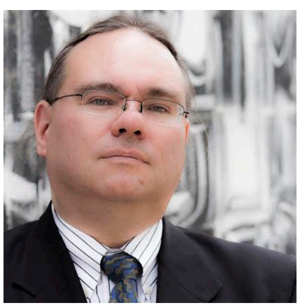
Every time we draw a new line, there is part of the figure to the left of the line, and part to the right of the line. (Strictly speaking, we need a directed line, with an arrow on it to define the forward direction, to pick left or right; but if our line isn't directed already, we can just pick a direction at random to make it directed.) Some regions are untouched by the new line, but others are cut into two parts, both colored the same. That is where problems are introduced. But if we could flip the colour of one of the regions, we'd be fine, except flipping the colour of one region causes a chain reaction. You should draw a few diagrams and see what happens. I recommend you "colour" the regions as "heads" and "tails" so you can use coins to easily change colour. Can you confine the chain reaction to one side of the line?

After some experimentation, you will probably notice that you can always consistently colour the regions of the square with just two colours after you add a new line. But how can we be sure? A little bit of thought should convince you that the following procedure always works: we draw a new line, which may divide some regions in two and leave others alone. If a region was heads before, and the line divided it in two, then each part would become heads. Then we take every coin on the left side of the line and flip it, changing colour of every region on the left side of the line. We leave every region's colour on the right side of the line alone. We have a map that is still colored with exactly two colours. (Why?)

The process works for every line that we add. So we know that two colours are enough for any such patchwork. Coupled with the knowledge that one colour is not enough for the vast majority of patchworks, we can conclude that the fewest number of colours with which we can colour any such arrangement is two.

If we were not ambitious with this problem, we might still be guessing the answer.

Here is another, harder problem along similar "lines": a simple, closed curve is a curve that ends at its starting point but otherwise doesn't intersect itself. For example, 0 is a simple, closed curve, while 8 is not. The shape of a simple closed curve can otherwise be arbitrarily complicated. On an empty square, we draw any number of simple, closed curves. The resulting figure will have many different irregularly shaped regions. What is the fewest number of different colours you need to colour any such arrangement?



*Edward Doolittle is Associate Professor of Mathematics at First Nations University of Canada. He is Mohawk from Six Nations in southern Ontario. He earned his PhD in pure mathematics at the University of Toronto in 1997. Among his many interests in mathematics are mathematical problem solving, applications of mathematics, and Indigenous mathematics and math education. He is also a champion pi-day debater at the University of Regina's annual pi day, taking the side of the other transcendental number, e.*

## Reflections

Reflections *is a monthly column for teachers, by teachers on topics of interest to mathematics educators: lesson plans, book/resource reviews, reflections on classroom experiences, and more. If you are interested in sharing your own ideas with mathematics educators in the province (and beyond), consider contributing to this column! Contact us at thevariable@smts.ca.*

### Summer "PD" opportunities – For kids!

*Sharon Harvey*

Each month, the SMTS publishes a list of upcoming PD for teachers. For this month's blog post, I thought I'd copy that—with a twist.

Summer is almost upon us, and most of us have our own summer professional development lined up (watch for my reflection from Twitter Math Camp later this summer). Summer PD is amazing! I know it's supposed to be our time "off," but with the hustle and bustle of the school year, great PD can be difficult to attend for many reasons. The number one advantage for me is that summer PD doesn't take me out of my classroom!

With this idea in mind, I thought I would put together a list of some summer "PD" options for your students (or your own kids). Summer camps have been a life saver in my own house—I mean, we all love and adore our children, but 8 weeks is a long time, folks. We have sent our own kids to many of the camps listed below, and they have raved about all of them. Now, this list is in no way comprehensive (there are a lot of summer camp options out there!), but I feel confident that the options below will be an excellent experience and will help keep children's minds active! Please note that I did not include Activity Camps. They are also fabulous, but the list can only be so long!

**Engineering for Kids**
Regina, Saskatoon

Engineering for Kids offers summer programs, as well as programming through the year, and their camps were one of the biggest hits among my students and my own kids. Sign up a kid, and you might find something useful for your own classroom here! They have programs for ages 4 to 16 that range from Pirate Academy to Engineering of Food to Video Game Design.

Regina: http://engineeringforkids.net/location/regina/summercamps
Saskatoon: http://engineeringforkids.net/location/saskatoon

**SCI-FI Summer Camps**
Saskatoon, Yorkton, Prince Albert, North Battleford

SCI-FI was always a hit in our house as well. Not only did we send our kids there, my husband spent many summers working with the program as well. SCI-FI is an established program and you can be sure that your little ones with have a consistently amazing time. There are camps for students in Grade 1 and up, with camp themes ranging from Science, Technology, and Computer Science to Medical Science and VetMed (and more). Older kids? Consider having them become a CIT (counsellor in training) for one of the camps. This year, camps are being offered in Saskatoon, Yorkton, Prince Albert, and North Battleford.

See http://scifi.usask.ca/summer-camps/index.php

**Sasktel Summer Cool Camps at Royal Saskatchewan Museum**
Regina

This camp came highly recommended by one of my Grade 11 students. I did a quick survey of the website and it looks like a camp full of adventure and learning! There are a number of camps to choose from for students entering Grades 1-5. They have single day camps, weeklong day camps, and even an adventure camp that heads out to Eastend for a weeklong adventure.

See http://www.royalsaskmuseum.ca/programs/public-programs-in-regina/sasktel-summer-cool-camps

**Saskatchewan Science Centre Summer Day Camps**
Regina

I've taken kids on numerous field trips to the Science Centre in Regina, and they always have an awesome time. A week at the Science Centre sounds like an even more awesome time. They offer camps for kids aged 6 – 11 and four camp themes: Story Science, The Great Outdoors, Innovate 101, and Science Surprise.

See http://www.sasksciencecentre.com/daycamps/

**Debate Summer Camp**
Saskatoon, Regina

Tired of arguing with your pre-teen? Give them the chance to learn how to argue among themselves. This camp teaches students the basics of structured debate and attendees will learn how to structure and defend *both* sides of an argument. The weeklong day camp is open to students in Grades 5 – 12 and is offered in Saskatoon and Regina.

See http://www.saskdebate.com/special-events/camp

**BONUS: KinderBuzz Webiste**

If you are in the Regina area, I suggest you take a look at KinderBuzz. It has compiled event calendars, summer camps, and other events happening around the city throughout the school year. The link below will take you to their listing of summer camps, but there is so much more information on this website!

See http://www.kinderbuzz.com/features/summer-camps-and-around-regina

There is no way that I could make a list of all of the options. There are just too many. But hopefully this list will help you find something that will keep your students' or your own kids' minds active during the summer, at least for a little while.

If you are aware of other exciting summer opportunities, please share them with us! Comment below or send an email to thevariable@smts.com.

*Sharon Harvey has been a teacher within the Saskatoon Public School Division for eight years. She has taught all secondary levels of mathematics, as well as within the resource program. She strives to create an inclusive and safe environment for her students.*

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## Intersections

*In this monthly column, you'll find information about upcoming math (education)-related workshps, conferences, and other events that will take place in Saskatchewan and beyond. If travel is not an option at this time or if you prefer learning from the comfort of your own home, see the Online workshops and Continuous learning online sections. Some events fill up fast, so don't delay signing up!*

*For more information about a particular event or to register, follow the link provided below the description.*

### Within Saskatchewan

***Conferences***

**SUM Conference**
November 4th – 5th, Saskatoon, SK
Presented by the SMTS

Our own annual conference! Join us for two days packed with learning opportunities, featuring keynotes Max Ray-Riek and Grace Kelemanik and featured presenter Peg Cagle. This conference is for math educators teaching in K-12, and registration includes lunch on Friday and a two-year SMTS membership. See the poster on page 3 for more information, and keep checking the SMTS website (www.smts.ca) in the coming months for registration details.

*Presenters:* Are you interested in presenting at SUM 2016? The SMTS is now accepting proposals for one-hour sessions focused on improving the teaching and learning of mathematics. Presenters receive one complimentary registration (includes lunch and a 2-

year membership). Head to www.smts.ca/sum-conference/sum-call-for-proposals/ to submit your proposal.

*Workshops*

**Educator Well-Being: You First**
July 26th, Moose Jaw, SK
Presented by the Saskatchewan Professional Development Unit

"In case of an emergency, put your own oxygen mask on first, before assisting others." While we understand the value of this familiar phrase, doing the equivalent in our busy lives is a challenge. Seldom do we put our own well-being first. This is especially true for educators – the caring profession. Yet, the benefits of well-being are recognized by research and by testimonials from people all over the world.

This workshop is based on Positive Psychology, an emerging field of scientific study and research that promotes optimal human functioning. Well-being is a skill to be learned that requires practice, just like math, athletics and playing an instrument.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/educator-well-being-you-first

**Comprehension Strategies in All Subject Areas**
July 27th, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Students are faced with increasingly complex texts in every subject area. Research literature confirms the importance of explicitly teaching comprehension strategies to students to support their understanding. By explicitly teaching comprehension strategies in subject areas such as science and math, teachers can help students develop deeper understanding of these and other subject areas. This workshop will have participants experience a number of practical strategies that they can connect back to the subjects that they teach.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/comprehension-strategies-all-subject

**Structures for Differentiating Elementary Mathematics**
July 28th, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

We know through formative assessments that our elementary students are at different places in their understanding of mathematics, but how do we structure our classrooms to meet their individual needs? This workshop will provide the opportunity for participants to design their classroom structure so that it allows children to move flexibly among large groups, small groups and individual instruction. By having a structure in place, teachers can create a differentiated learning experience without creating individualized learning programs for every child.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/structures-differentiating-elementary

**Multi-Graded Mathematics Instruction**
August 3rd, Saskatoon, SK
$110 (early bird)
Presented by the Saskatchewan Professional Development Unit

How do you address all of the needs within your combined grades mathematics classroom? By looking at themes across curricula, teachers can plan for diverse needs and address outcomes at two grade levels without having separate lesson plans. Curricular through lines and planning templates will be shared that are helpful for identifying how concepts grow over the grades, so that you can build a learning continuum within your instruction.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/multi-graded-mathematics-instruction-0

## Beyond Saskatchewan

**Waterloo Math Teachers' Conference**
August 23rd – 25th, Waterloo, ON
Presented by the Centre for Education in Mathematics and Computing

A conference for teachers of grade 7-12. While the Grade 9-12 sessions are directed towards university preparation and mainly Ontario teachers, teachers from any province or country will benefit as well. Registration is now open and spots fill up fast, so sign up early! Participation is restricted to two teachers per school.

See http://www.cemc.uwaterloo.ca/events/mathteachers.html

## Online Workshops

**Math Daily 3**
July 3th–July 30th
Presented by the Daily CAFÉ

Learn how to help your students achieve mathematics mastery through the Math Daily 3 structure, which comprises Math by Myself, Math with Someone, and Math Writing. Allison Behne covers the underlying brain research, teaching, and learning motivators; classroom design; how to create focused lessons that develop student independence; organizing student data; and differentiated math instruction. Daily CAFE online seminars combine guided instruction with additional resources you explore on your own, and are perfect for those who prefer short bursts of information combined with independent learning.

*The seminar includes:*

- online access to videos, articles, and downloadable materials
- access to an exclusive online discussion board with colleagues
- a certificate of attendance for 15 contact hours

See https://www.thedailycafe.com/workshops/10000

## Continuous Learning Online

**Education Week Math Webinars**
Presented by Education Week

Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling and Differentiation.

Past webinars: http://www.edweek.org/ew/webinars/math-webinars.html
Upcoming webinars:
http://www.edweek.org/ew/marketplace/webinars/webinars.html

## Call for Contributions

Did you just deliver a great lesson? Or maybe it didn't go as planned, but you learned something new about the complexities of teaching and learning mathematics. Maybe you just read a book or attended a workshop that gave you great ideas for presenting a topic your students have always found difficult, or that changed your perspective about some aspect of teaching. **Why not share your ideas with other teachers in the province – and beyond?**

*The Variable* is looking for a wide variety of contributions from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants and coordinators, teacher educators, researchers, and students of all ages. Consider sharing a favorite lesson plan, a reflection, an essay, a book review, or any other article or other work of interest to mathematics teachers in Saskatchewan. If accepted for publication, your piece will be shared, as part of this periodical, with a wide audience of mathematics teachers, consultants, and researchers across the province, as well as posted on our website.

**We are also looking for student contributions**, whether in the form of artwork, stories, poems, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom and their school, and for teachers to recognize their students' efforts during their journey of learning mathematics.

All work will be published under a Creative Commons license. If you are interested in contributing your own or (with permission) your students' work, please contact us at thevariable@smts.ca.

We look forward to hearing from you!