Saskatchewan Mathematics Teachers' Society

AFFILIATE
NCTM
NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

In this issue

Cover Image

This month's cover photo was taken at the National Music Centre (NMC) in Calgary, Alberta. The NMC has won numerous architectural design awards. According to Brad Cloepfil, lead architect, inspiration for the building was drawn from Canada's iconic landscapes—"from the cadence of waves to the lullaby of lakeshores, from the silence of the prairies to the echo of the Arctic, and the energy and diversity of Canada's urban spaces." For more information, see www.studiobell.ca.

SMTS Executive (2016-2018)

Notice to Readers

The views expressed in *The Variable* are those of the author(s), and not necessarily those of the Saskatchewan Mathematics Teachers' Society. In the publishing of any works herein, the editorial board has sought confirmation from all authors that personal quotes and examples have been included with the permission of those to whom they are attributable.

Notice to Contributors

The Variable welcomes a variety of submissions for consideration from all members of the mathematics education community in Canada and beyond, including classroom teachers, consultants, teacher educators, researchers, and students of all ages, although we encourage Saskatchewan teachers of mathematics as our main contributors. Submitted material is assessed for interest, presentation, and suitability for the target audience.

Please submit articles by email to thevariable@smts.ca in Microsoft Word format. Articles should be limited to 3000 words or less; authors should also include a photo and a short biographical statement of 75 words or less. Accepted contributions are subject to revision. Editors reserve the right to edit manuscripts for clarity, brevity, grammar, and style.

Call for Contributions

The Variable **is looking for contributions from all members of the mathematics education community**, including classroom teachers, consultants, and teacher educators. Consider sharing a favorite lesson, an essay, a book review, or any other work of interest to mathematics teachers in Saskatchewan. When accepted for publication, your article will be shared with a wide audience of mathematics educators in Saskatchewan and beyond.

We also welcome student contributions in the form of artwork, stories, interesting problem solutions, or articles. This is a great opportunity for students to share their work with an audience beyond that of their classroom, and for teachers to recognize students' efforts during their journey of learning mathematics.

All work is published under a Creative Commons license. **To submit or propose an article, please contact us at** thevariable@smts.ca. We look forward to hearing from you!

Ilona & Nat,
Editors

Message from the President

Happy November!

When we first started publishing *The Variable* and I was trying to write a monthly president's message, it never once occurred to me that I would ever have too much to say—yet here we are!

So first, and, I'd argue, most importantly: Thank you! Thank you to everyone who came out and made this year's Saskatchewan Understands Mathematics (SUM) Conference extra special. It was particularly exciting to see the number of school teams, as well as administrators and superintendents, learning together. There is great value in shared learning experiences, as these can serve as catalysts for sustained learning opportunities. I hope you'll share with us—on Twitter, or in *The Variable*—how your team is using your shared learnings from SUM Conference to move your teaching forward. Myself, I'm still processing and reflecting on what I learned during the two days, and thinking about what my short- and long-term goals will be in my work. Steve Leinwand, our keynote speaker, reminded us to pick just two things, but that seems easier said than done!

This brings me to my next topic: Math is in the spotlight! Specifically, the teaching and learning of mathematics and the mathematics curriculum. Now, Egan Chernoff would be right to remind us that this is not the first time math has caught some limelight (for a complete recent history, you can peruse his blog), but the recent mention in the throne speech feels different—possibly because this time, the media is taking care to seek out teachers' perspectives on the issue, and that's something to celebrate in and of itself.

This rare opportunity to share our vision for mathematics education in Saskatchewan is one we should be grateful for. Of course, I will admit that at the end of a long day, the last thing I want to do is explain to someone that yes, our curriculum does, in fact, require that students learn their multiplication tables, or that the textbook is not a curriculum, but a resource, one of many that teachers may choose to use. But—but!—I need to remind myself to be grateful that we are talking about mathematics education at all, and that these small conversations with our family and friends are actually very important to the big picture. We need to have these small conversations before we can move on to the bigger ones that are looming in our future.

I hope you will join me in sustaining these conversations about mathematics education, big and small, around the province. While we may run into some tension, as with most things, I believe we need to start by finding common ground. What do we hope for our students? What do our teachers need in order to make these hopes a reality? And, above all, how do we put pressure on those best able to provide these supports, supports that are backed by research, and not opinion or hearsay? This will be no small task, but it deserves our time. Our students are worth it, and we can certainly all agree on that.

Michelle Naidu

Problems to Ponder

Welcome to this month's edition of Problems to Ponder! Have an interesting solution? Send it to thevariable@smts.ca for publication in a future issue of The Variable!

Primary Tasks (Kindergarten-Intermediate)

Building Skyscrapers of Different Heights[1]
Stack 10 blocks (or linking cubes) to make any number of towers. The heights of the towers must all be different. Can you find all of the solutions?

Adaptations and extensions: Stack 6 blocks to make towers of different heights. Stack 15 blocks to make towers of different heights. Find all of the solutions.

Get to Zero[2]
This game, played in teams of 2 or 3, provides students practice subtracting from 999. Students should be encouraged to check each players work and provide feedback for mistakes.

1. First, on a sheet of paper, each player writes the players' names and the number 999 under them.

---

[1] MathPickle. (2010). Building skyscrapers of different heights. Retrieved from
   http://mathpickle.com/project/4112/
[2] Get to zero. (n.d.). Retrieved from https://www.youcubed.org/tasks/get-to-zero/

2. A player rolls three dice, then arranges the three numbers (for example, 2, 3, 5) in some order (for example, 235, 352, 532, and so on) and subtracts that 3-digit number from 999.
3. The players take turns, rolling the die to make a number and subtracting that number from their total.
4. The winner is the first player to reach 0, but they must get to 0 exactly.
5. At any time, a player may choose to roll only one or two dice, instead of three dice. If the only numbers a player can make are larger that his remaining score, the player loses his turn.

Adaptation: Start with the number 99 and two dice.

Ice Cream Scoop [3]
In shops with lots of ice-cream flavors there are many different flavor combinations, even with only a 2-scoop cone. With 1 ice-cream flavor there is 1 kind of 2-scoop ice cream, but with 2 flavors there are 3 possible combinations (eg vanilla/vanilla, chocolate/chocolate, and vanilla/chocolate).

Extensions:
- How many kinds of 2-scoop cones are there with 3 flavors? 4 flavors? 5 flavors?
- Explain how you would find the number of combinations for any number of flavors.

Intermediate and Secondary Tasks (Intermediate-Grade 12)

I Win![4]
Two players each roll an ordinary six-sided die. Of the two numbers showing, the smaller is subtracted from the larger. If the difference is 0, 1, or 2, Player A gets 1 point. If the difference is 3, 4, or 5, Player B gets 1 point. The game ends after 12 rounds. The palyer with the most points wins the game. Is this game fair?

Extensions:
- Can you change the rules of the game so that it is fair?
- Roll two standard six-sided dice and divide the larger number showing by the smaller number. Player A gets 1 point if there is a remainder other than 0. Player B gets 1 point if the remainder is 0. Is this a fair game? If not, how would you make it fair?
- Assign numbers to the faces of the two dice so that, on any one roll if you add the numbers on the top faces, each sum from 1 to 12 is equally likely.

---

[3] *Adapted from* Ice cream scoop. (n.d.). Retrieved from https://www.youcubed.org/tasks/ice-cream-scoop/
[4] National Council of Teachers of Mathematics. (2004). Math challenge #26: I win! Retrieved from http://figurethis.nctm.org/challenges/c26/challenge.htm

Domino Tilings[5]

How many ways are there to tile a 2×5 rectangle with 1×2 dominoes? Here are two possibilities:
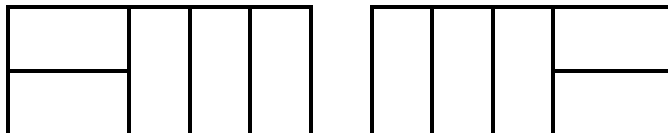
Extensions:

- How many ways are there to tile a 2×4 rectangle with 1×2 dominoes? a 2×6 rectangle? Explore for $2 \times n$ rectangles. Once you have a conjecture, try to explain why it is true.
- How many ways can you tile a $1 \times n$ board with 1×1 squares and 1×2 dominoes?

Cut the Cake[6]

You and five friends want to share a 9-inch square chocolate cake with marshmallow icing. How can you cut the cake so that each person receives an equal share of both cake and icing? (Don't forget the icing on the sides!)

Extensions:

- How would you share a 9-inch square cake among 5 people so that each person receives an equal share of both cake and icing?
- In how many different ways can you divide a rectangle into two pieces of the same size and shape?

---

[5] Su, F. E. et al. (n.d.). Domino and square tilings. Retrieved from https://www.math.hmc.edu/funfacts/ffiles/20003.4.shtml

[6] *Adapted from* National Council of Teachers of Mathematics. (2004). Math challenge #54: Cut the cake. Retrieved from http://figurethis.nctm.org/challenges/c54/challenge.htm

Alternate Angles

Alternate Angles is a bimonthly column on problems from multiple perspectives: various methods that could be used to solve them, insights we get from their solution, the new paths that they can lead us to once they have been solved, and how they can be used in the classroom.

❖

Radical Radicals

Shawn Godin

Welcome to Alternate Angles! In my article "Mining Math Contests for Good Problems," which appeared in the last issue of *The Variable*, I looked at many different ways that problems from math contests could be used in the mathematics classroom. This column, however, will take a slightly different approach. In each installment of Alternate Angles, we will look at a single problem in detail: its solution(s), its extensions, and ways that it can be used in the classroom. As mathematics is not a spectator sport, at the end of each column, I will present the problem that will be the focus of the next issue. This will give you the chance to play with the problem and compare your thoughts with mine. I that hope you will enjoy this column, and I welcome any feedback you have.

Now, let's get on with the fun! At the end my article in the last issue, I left you with the following problem to contemplate:

Evaluate $\left(\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}} \right)^2$.

This problem was from the 2014-2015 Nova Scotia Math League, Game 3 (problem A from the pairs relay). At first glance, it may not seem like much of a problem. We can type it into a calculator and get 6, and we are done. At this level, the expression is nasty enough that the problem is at least a good exercise in calculator use.

However, during the contest, students were not allowed to use calculators. This would force students to use properties of radicals and the distributive property in order to evaluate the expression. Hopefully, some students will recognize the similarity to squaring a binomial. Thus, we get

$$\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)^2 = \left(\sqrt{2+\sqrt{3}}\right)^2 + 2\left(\sqrt{2+\sqrt{3}}\right)\left(\sqrt{2-\sqrt{3}}\right) + \left(\sqrt{2-\sqrt{3}}\right)^2$$

In the first and last terms, we can just lose the square root signs, leaving us with $2+\sqrt{3}$ and $2-\sqrt{3}$, which sum to 4. The middle term, after using some properties of square roots, will hopefully remind us of a difference of squares, giving us

$$2\left(\sqrt{2+\sqrt{3}}\right)\left(\sqrt{2-\sqrt{3}}\right) = 2\sqrt{(2+\sqrt{3})(2-\sqrt{3})} = 2\sqrt{4-3} = 2.$$

Adding this to 4, we find that the final result of our problem is 6, confirming what we discovered using a calculator.

The problem, then, could be used as a calculator exercise or to reinforce some important algebraic identities. But the real fun begins when the problem is done, and you realize that the problem has shown us that

$$\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}} = \sqrt{6}.$$

That is, just like $\frac{2}{3}$ is the reduced or a simplified form of the fraction $\frac{18}{27}$, $\sqrt{6}$ is a simplified form of $\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}$. It begs the question: What other radical expression of this form exist?

A first conjecture might come from the fact that $2\times3 = 6$, so maybe the result is simple. Unfortunately, choosing numbers randomly, like 10 and 7, does not yield the desired result. That is,

$$\left(\sqrt{10+\sqrt{7}}+\sqrt{10-\sqrt{7}}\right)^2 \approx 39.3 \neq 10\times7$$

Maybe, then, the result we obtained was unique. We could keep trying expressions of this form at random, hoping for a desirable result, or we could employ some technology to help us out. A spreadsheet would be a great tool to evaluate our "form" for various values, and I would encourage you to give this a try.

I will instead use the online graphing calculator Desmos, available at www.desmos.com. If you have never used it, I strongly suggest you check it out. In addition to being a free, intuitive graphing calculator, it also works as a regular calculator and allows you to calculate or graph expressions involving parameters that you can manipulate using sliders. In this case, entering the expression with variables a and b prompts Desmos to ask if you would like to create sliders (see Figure 1).

$$\left(\sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}\right)^2$$

add slider: a b all

Figure 1

If you click, *a*, *b*, or all, sliders will be created that will control the value of the variables. In Figure 2, the slider for *a* is displayed. By default, the value of the variable will be kept between −10 and 10, as shown. Clicking the "play" button, located on the left, will cause the value of the variable to change continuously through the values in the given range.

$a = 1$

−10 ———————•——————— 10
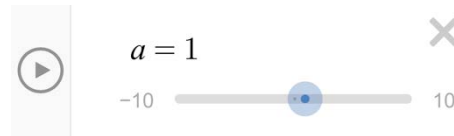
Figure 2

You can adjust the range of values and the difference between successive values by clicking on either of the numbers. In this case, I have set the step size to 1, which constrains the value of *a* to the integers, and specified that the value of *a* should be between 1 and 25 inclusive (see Figure 3).

$a = 1$

1   $\leq a \leq$ 25   step: 1
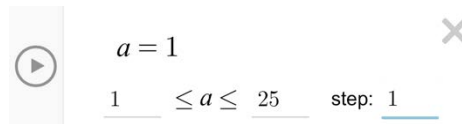
Figure 3

We can constrain the values of *b* similarly. Then, when we adjust the values of the variables using the sliders, we can see the result in the original box. In Figure 4, we can see the values from the original problem used on the left, and our "guess" that didn't work on the right.

$$\left(\sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}\right)^2$$ = 6

$a = 2$

1 •——————————————— 25

$b = 3$

1 —•—————————————— 25

$$\left(\sqrt{a+\sqrt{b}} + \sqrt{a-\sqrt{b}}\right)^2$$ = 39.287301522

$a = 10$

1 ———————•———————— 25

$b = 7$

1 —————•—————————— 25
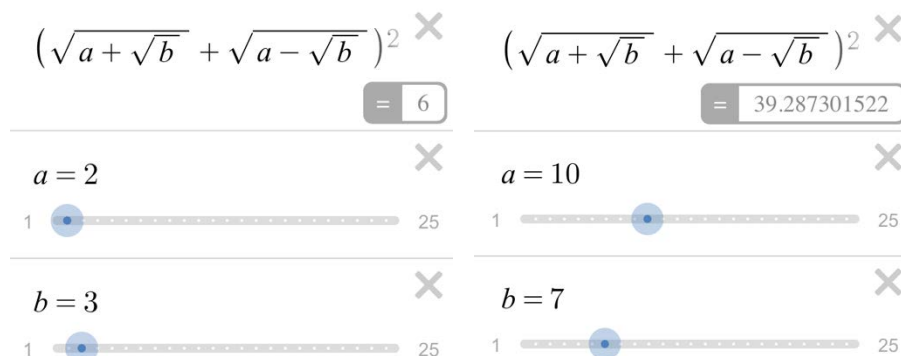
Figure 4

We can now play with the values of a and b and look for values of our expression that are whole numbers. If we go to the top left part of the screen and click on the + sign, we see (Figure 5) that Desmos allows us to enter expressions, notes (for giving instructions to people looking at the page), tables of values, folders (for organizing your content) and images.

Figure 5

After trying all integer values of our variables with $1 \leq a, b \leq 25$ and putting them into a table of values, we get the graph shown in Figure 6. (The settings of the graph can be adjusted by clicking on the graph settings icon, 🔧, located in the top right corner of the screen.)
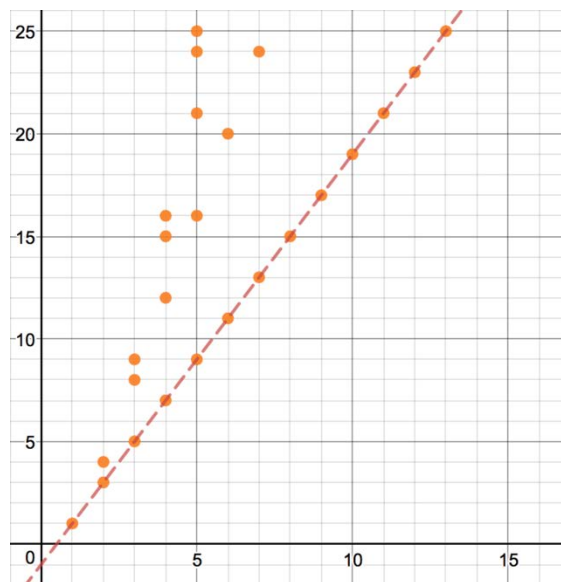
Figure 6

One pattern that immediately jumps out at us is that the "lowest" points all seem to line up. The equation of this line is $y = 2x - 1$, shown as a dashed line in the figure. What does that tell us? Since we chose x and y to represent a and b, it means that if we have $b = 2a - 1$,

we should have a solution. How could we check this hypothesis? We could create another expression, $\left(\sqrt{A + \sqrt{2A - 1}} + \sqrt{A - \sqrt{2A - 1}}\right)^2$, and use Desmos to see what happens when we change the value of A (I have used a capital A to differentiate it from the lower case a we used earlier). If you do this, you will get only integer values. This leads us to suspect that all of these values work. Unfortunately, having a lot of cases that work does not mean that *all* values work. To show that this expression is always an integer, we need to evaluate the expression algebraically. Expanding and simplifying gives us

$$\left(\sqrt{A + \sqrt{2A - 1}} + \sqrt{A - \sqrt{2A - 1}}\right)^2 = 4A - 2,$$

which means that if we choose A to be a positive integer, $2A - 1$ will also be a positive integer, and, more importantly, so will $4A - 2$. If we choose $A = 2$, it yields the original problem (try it out). If we wanted to be more rigorous, we would also have to show that all of the radicals are real values. Indeed, if A is a positive integer, then $2A - 1 > 0$, so $\sqrt{2A - 1}$ is real. We also need $A - \sqrt{2A - 1} \geq 0$, which is true for $A \geq 1$, so all positive integer values of A generate a valid solution.

We have thus found an infinite family of solutions to our problem. However, if we check the graph in Figure 6, we see that this family does not contain all integer solutions. A little exploration will yield some other linear relations. If we look a little closer, we can find some parabolas as well. Interestingly, we noticed that our family of solutions was the set of "lowest" points. Once those are removed, the next set of "lowest" points gives us another linear family. If continue this way from the bottom up, we will continue to find new linear families of solutions. Alternatively, if we look at the "highest" points, they lie on a parabola. If we continue from the top down, we will find more quadratic families of solutions. It turns out, then, that we have an infinite family of lines, each containing an infinite number of solutions to our problem. Similarly, there is an infinite family of parabolas containing an infinite number of solutions.

You can follow the above suggestions to explore this problem in Desmos yourself, or you can access my Desmos graph at www.desmos.com/calculator/t1d9rw1jbi. I have added a few things, as well as some comments explaining some of the features. The content is organized in folders corresponding to various parts of this column.

Of course, we could also try to solve this problem algebraically. If we start with $\left(\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}\right)^2$ and simplify, we get

$$\left(\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}\right)^2 = 2a + 2\sqrt{a^2 - b}.$$

Then, if we want our result to be an integer (and real!), we need $a^2 - b = k^2$ for some non-negative integer k. We can rearrange this to get $b = a^2 - k^2$, which tells us for each non-negative integer k and for each positive integer a such that $a^2 - k^2 \geq 0$, $\left(\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}\right)^2$ is a positive integer.

If we look at this a case at a time, we can see what is happening. If $k = 0$, we get $b = a^2$, which means that if (a, b) is a *lattice point* (i.e., a and b are integers) in the first quadrant on the parabola with equation $y = x^2$, then $\left(\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}\right)^2$ is a positive integer. This turns out to be a trivial case, since we would have

$$\sqrt{a - \sqrt{b}} = \sqrt{a - \sqrt{a^2}} = \sqrt{a - a} = 0$$

and, similarly,

$$\sqrt{a + \sqrt{b}} = \sqrt{a + \sqrt{a^2}} = \sqrt{a + a} = \sqrt{2a}.$$

Hence,

$$\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}} = \sqrt{2a},$$

and the property is trivially true.

If $k = 1$, we get $b = a^2 - 1$, which means that if (a, b) is a lattice point in the first quadrant on the parabola with equation $y = x^2 - 1$, then $\left(\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}\right)^2$ is a positive integer. Similarly, if (a, b) is a lattice point in the first quadrant on the parabola with equation

$$y = x^2 - k^2$$

for any integer k, then $\left(\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}\right)^2$ is a positive integer. This describes the infinite family of parabolas with infinitely many solutions on them.

Having also seen an infinite family of lines, we can go looking for them as well. We previously showed that $b = a^2 - k^2$, and since we want real solutions, we need $b > 0$ (we will ignore the trivial case where $k = 0$). Without loss of generality, we can assume that $k > 0$.

Factoring the expression for b yields $b = (a - k)(a + k)$. If we introduce a new parameter, ℓ, such that $\ell = a - k$, then $k = a - \ell$ and we can rewrite the expression as

$$b = \ell(a + a - \ell) = 2a\ell - \ell^2,$$

which means that if (a, b) is a lattice point in the first quadrant on the line with equation

$$y = 2\ell x - \ell^2$$

for any positive integer ℓ, $\left(\sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}\right)^2$ is a positive integer.

In short, what started out as a calculation that could be handled with a calculator led to an investigation of infinite families of lines and parabolas whose positive-integer-valued points yield expressions of the same form as that of the original question. Depending on what your students know, there are still many things that they could discover from this problem. However, strongly-developed algebraic skills are not a prerequisite for exploring this problem—even younger students can discover the linear families using Desmos and verify the results numerically.

Now, it is time for some homework! In the next issue, I will be looking at the following problem:

> You have two hourglasses: One that measures 9 minutes and one that measures 13 minutes. Determine how to measure 30 minutes using these two hourglasses.

Take some time to play around with the problem or, better yet, offer it to your students. Until next time, happy problem solving!

❖

Shawn Godin teaches and is a department head at Cairine Wilson Secondary School in Orleans, Ontario. He strongly believes in the central role of problem solving in the mathematics classroom. He continues to be involved in mathematical activities: leading workshops, writing articles, working on local projects and helping create mathematics contests.

Spotlight on the Profession
In conversation with Susan Milner

In this monthly column, we speak with a notable member of the mathematics education community about their work and their perspectives on the teaching and learning of mathematics. This month, we had the pleasure of speaking with Susan Milner.

Susan Milner taught post-secondary mathematics in British Columbia for 29 years. For 11 years she organised the University of the Fraser Valley's secondary math contest, where her favourite part was coming up with post-contest activities for the participants. In 2009 she started Math Mania evenings for local youngsters, parents and teachers. This was so much fun that she devoted her sabbatical year to adapting math/logic puzzles and taking them into K-12 classrooms. Now retired and living in Nelson, BC, she is still busy travelling to classrooms and giving professional development workshops. In 2014 she was awarded the Pacific Institute for the Mathematical Sciences (PIMS) Education Prize.

◆

First things first, thank you for taking the time for this interview!

Thank you so much for inviting me to participate—I love talking shop!

One of your passions in life has been to enhance public awareness and appreciation of mathematics—a passion that has led you to develop and become involved in a wide variety of outreach activities, including workshops, classroom visits, and public events, for which you received the Pacific Institute for the Mathematical Sciences (PIMS) Education Prize in 2014. As Alejandro Adem, a former PIMS Director, remarked, "Susan Milner is an outstanding educator, who has worked tirelessly to share the joy of mathematics with countless students and teachers in BC" (PIMS, 2014).

What drew you to study, and then to teach, mathematics in your younger years? What fuels your outreach work today?

All the way through school, math always seemed like fun, especially when I learned something new. Those *aha!* moments are really quite addictive. I remember loving to figure out patterns for myself—staring with the times table in Grade 3, re-inventing parts of

modular arithmetic so I would know how at what time I had to go to bed in order to get the right amount of sleep, and teaching myself function notation in order to write math contests.

To me, the most appealing aspect of mathematics is that people can reason with each other, without resorting to emotional brow-beating, and without appealing to authority, power, or wealth. From the time I was 10 years old, Mr. Spock has been my hero, not because of how much he knows, but because of how clearly he lays out arguments.

> "To me, the most appealing aspect of mathematics is that people can reason with each other, without resorting to emotional brow-beating, and without appealing to authority, power, or wealth."

At university, I first studied philosophy and classics, taking calculus courses just for fun during my first degree. I thought about doing graduate work in philosophy, logic in particular, but that seemed impractical, so after I finished a degree in mathematics, off I went to do more math. Of course mathematics is more practical than philosophy, right? I ended up doing logic in topos theory, an abstract branch of category theory, which is in turn a highly abstract branch of mathematics. So much for practicality!

I then fell into teaching. When I took what was supposed to be a year off after my Master's degree to teach at Okanagan College, I found that I thoroughly enjoyed working with adults. It was exciting to try to help them make sense of complicated ideas, to watch light bulbs going on, and to watch students get excited about what they were learning. That was always the best part of teaching, at any level, in any course.

Now that I have retired from teaching, I get to focus exclusively on having fun with students of all ages. I love watching them get caught up in a new game, figuring things out, and developing their abilities to think logically. I also really enjoy the excitement many teachers express as they watch their students get highly engaged and demonstrate abilities that they didn't know they had.

One of the main ways you have sought to share the joy of mathematics is by introducing children and adults of all ages to a variety of mathematical puzzles and games [see www.susansmathgames.ca, *as well as Issues 1(3), 1(5), and 1(8) of* The Variable]. *You have also suggested that, perhaps, "puzzles and games can play a small part in changing the Canadian pattern of having the math-averse and math-fearful pass on their issues to the next generation" (Milner, 2013, p. 12).*

The question begs to be asked: What do puzzles and games have to do with mathematics?

The games I like involve logic, not chance and not knowledge. The types of reasoning we use in these games are exactly the types of reasoning that mathematicians use all the time to explore ideas and prove results. For example: "If A is true, then B must follow," "If C were true, then D couldn't be true, but D must be the case, so C cannot hold," "If E were true, then there would be no way for us to achieve F, so E can't be true," "G is true if and only if H is true"... Don't those abstract statements make your head spin? Yet I have heard many, many children say things exactly like that in their own words, while they were trying to work out some part of a puzzle.

Some games involve a lot of spatial reasoning, both two-dimensional and, occasionally, three-dimensional. Shape (geometry) is often sadly neglected in the school curriculum, particularly in the middle and high school years, yet it is a huge aspect of the advanced mathematics we need to model our world. It's also highly appealing to most people, being tied so closely to the physical world. If we can explore reasoning using that appealing physicality, we are likely to reach far more students than we would otherwise.

> "It is essential to be aware of the difference between what we know for sure and what we hope, guess, or want to be true. I think this is essential for sensible discussion in all aspects of life, not only in mathematics."

Beyond that, playing with these puzzles encourages what have come to be called "mathematical habits of mind," which include, among other attributes, persistence, attention to detail, and willingness to start over.

One of the most necessary aspects of all of the games and puzzles I take into classrooms is that it is essential to be aware of the difference between what we know for sure and what we hope, guess, or want to be true. I think this is essential for sensible discussion in all aspects of life, not only in mathematics.

And what do puzzles and games have to do with "school" mathematics—that is, the mathematics that students learn in elementary and secondary schools?

Most of the games I first introduce to classes use nothing beyond counting, two-digit arithmetic, and awareness of shape and colour. The reasoning, however, can be surprisingly sophisticated.

There are some puzzles you can use if you want students to practice some basic skill. For example, the Rectangles puzzle relates shape to the factoring small numbers, Kakurasu is terrific for adding and subtracting, and Mathdoku (KenKen) is good for arithmetic of small numbers. As far as I am concerned, though, in each case, the puzzle has to be interesting enough that the "good for you" aspect is all but invisible.

I don't know about Saskatchewan, but in British Columbia the curriculum has recently started to include more of an emphasis on pattern recognition, reasoning, and puzzle-solving. I've been delighted with that, as it makes it easier for teachers to point to the learning outcomes they are meeting when they use my materials.

As you share on your website and in Milner (2013), even students who have had unpleasant experiences in "formal" mathematics courses typically enjoy mathematical puzzles. In some cases, you have found that opportunities to work on puzzles increased both confidence and skills in the subject among students who didn't typically see themselves as "math brains." Why might this be so?

All of us have had experiences where we've failed at something. If we've failed at it several times, many of us tend to be very wary of putting ourselves in the position to fail again, and even if we do try, maybe with encouragement from someone else such as a teacher, we get so anxious that we can't function properly. There seems to be no way to break out of that downward spiral.

It's difficult to change one's picture of oneself, especially if negative aspects have been reinforced over the years; however, I think that "something completely different" can be a huge help. When a game or puzzle catches the attention of people who've had a rough time with math, they often seem able to break the cycle of negative self-talk because this doesn't feel like the math that has caused them so much anxiety. Nearly all of my most math-anxious pre-teaching students found a type of puzzle or two that they could solve easily, after which they became hooked and sought out harder puzzles. Once they succeeded, they were of course eager to keep succeeding.

Also: We tend to be better at tasks we see as interesting, or at least we are more likely to focus on them. It seems that playing with patterns and solving puzzles is a very human activity—we just like playing games. The stakes are lower, so there is less to be anxious about. For the math-anxious, games that involve manipulatives are even better than those involving only pencil and paper, because people can just sweep away all evidence of their wrong answer.

On your website, you remark that "mathematics is important, hard work," but that "most mathematicians will admit that they think of their work as play" (Milner, n.d.). And in Milner (2013, p. 10): "Yes, mathematicians play all day!" Could you elaborate?

At some point during my years of teaching, it became very clear to me that the way mathematicians talk about problems is very different from the way many people imagine: "Let's play with it" comes up remarkably often, as does "that's an elegant solution." At the start of term, my students laughed at me when I talked like that, but if I'd done a good job in the classroom, they would eventually start using that language, too. I am not alone in thinking that having a playful spirit and appreciating beauty in what we do lightens everything up and frees us to be more creative.

> "At some point during my years of teaching, it became very clear to me that the way mathematicians talk about problems is very different from the way many people imagine."

One of the things I love about abstract mathematics is that we can take a (small) set of definitions and maybe an axiom or two, and build an incredibly far-reaching system; think of classical geometry. So what else is a game, but trying to accomplish something within an agreed-upon a set of constraints? The bonus is that mathematics is remarkably good at modelling the world—it is not uncommon for apparently outrageously abstract mathematics to turn out to be essential at some time in the future. Number theory is a good example: According to the mathematician G. H. Hardy in the early 1900s, it was the purest form of mathematics because it had no use whatsoever; in particular, it could not be used for war or commerce. Now we cannot live without number theory in our internet security, which is essential to both war *and* commerce.

Is it a mistake, then, that mathematics curricula have largely ignored the playful nature of mathematics, and the role of play in learning? (On your website, quoting Plato, you write: "Do not keep children to their studies by compulsion but by play" [Milner, n.d.].) If so, how might play be integrated into the teaching and learning of mathematics, beyond the primary grades?

Most definitely! It seems to me that most people work much harder at their games, hobbies, and sports than they do at what they consider their "real" work. Motivation makes a huge difference, as any teacher knows.

To start, our own fear and/or anxiety has to go out the door, as students pick up on that immediately. We need to model what we believe—if we are going to see where a train of reasoning leads us, let's actually follow it, not try to force it into a particular direction. (Think about how that connects to science, politics, everyday life!) If we end up in a mess, let's go back and figure out what happened.

I startled several of my colleagues by setting project problems in calculus that I had no idea of the answer to—as a class, we would agree on the problem and the ground rules, then I would set the students loose to figure it out and write up their best answers. They couldn't figure out what answer *I* wanted, because I didn't have one, so the game became more like actual applied mathematics: They had to present their arguments in a way that would convince me that they had a good solution to a messy problem.

And just to be clear, when I talk about playing, I'm not talking just about the gamification of learning, if that means turning it into a sequence of little steps with small rewards for succeeding in each one. That can be effective if it's not over-used, but playfulness comes in many forms and we are not all suited for all forms. You can probably come up with several widely-differing examples of playful teachers, conference presenters, and mentors in your life. Genuine enthusiasm is impossible to fake and impossible to resist, so I think the trick may be to figure out what really excites *you* about a particular topic or problem, and then… to play with it.

> "Genuine enthusiasm is impossible to fake and impossible to resist, so I think the trick may be to figure out what really excites *you* about a particular topic or problem, and then… to play with it."

Within two minutes of walking into a new classroom, I can sense if the teacher has a playful attitude towards mathematics. Sometimes the students will already have played some math/logic puzzles, but sometimes the playfulness has revolved around solving difficult or open-ended problems together. Students fortunate enough to have a playful teacher tend to focus quickly when faced with a new game: "What's the goal? What do we know?" The whole class is much more likely to participate in the doing-an-example-together stage of my introduction, being willing to take turns supplying answers and listening to each other. Not that they are all perfect angels, but I'd say that students who are used to playing math games of one sort or another together are more likely to respect each others' contributions than are students in a class that has focused solely on getting the right answer and moving on to the next question.

Teachers in the latter type of class are often very surprised by which of their students turn out to be good at spotting patterns or at thinking ahead several steps in a logical chain. I love it when that awful stereotypical distinction of "math-mind" versus "not a math-mind" gets broken, both in the mind of the teacher and in the minds of the students themselves. I was delighted and touched when a teenaged First Nations student said to me in the hallway an hour after class, "Thank you for bringing your games to our class. I really liked Towers. It's a smart person's game. It made me feel smart." I'd like to think that she learned something about her abilities and gained some confidence that day.

You share a wide variety of mathematical puzzles and games on your website, www.susansmathgames.ca, many of which you have shared over the years with K-12 students in classrooms or during workshops across British Columbia. But surely, there are puzzles that don't make your list.

I definitely try out more puzzles with classes than appear on the site. I put something up on the website only once I have enough classroom experience to be able to describe a reliable way of introducing the game and once I am sure that my introductory puzzles are graduated appropriately for students to move smoothly from one level to the next.

In your view, what makes a good *mathematical puzzle or game? Do you have a personal favorite?*

For classroom purposes, a "good" puzzle has simple rules, is visually appealing, starts out fairly easily, and progresses through several different levels. Students should be able to see themselves solving harder puzzles at the end of 20-25 minutes. I hope that they can experience a couple of "aha!" moments, even minor ones, because that leaves them excited and wanting to do more.

The best puzzles to start with are the ones I've shared in various editions of *The Variable* [*see Issues 1(3), 1(5), and 1(8) of* The Variable –*Ed.*]. Hidato is very easy to explain and everyone seems to enjoy it, at any age. Rectangles is great for any age from about mid-Grade 3 to adult. Towers is terrific from about Grade 5 or 6 and up, but you should start with manipulatives so that everyone can understand the rules. Kakurasu takes a bit of effort to figure out, but students from about Grade 7 and up have found it exciting. Set is wonderful for the many types of games you can play once you know "the rule."

While I find it exciting to learn a new game, I tend to use familiar games to clear my mind and help me relax. There are a few games I find absorbing enough to play regularly online: Kakurasu, Calcudoku (Kenken), Three-in-a-Row (Unruly), Magnets, Neighbours (Adjacent), and Kakuro. Lately, I've gotten into Nurikabe again, and I've really enjoyed Slant and Futoshiki (Unequal). All of these appear on the BrainBashers site (www.brainbashers.com) and/or Simon Tatham's Portable Puzzle Collection (www.chiark.greenend.org.uk/~sgtatham/puzzles/). Set can also be played online (e.g., at www.setgame.com/set/puzzle) and clears the mind remarkably well.

Thank you, Susan, for taking the time for this conversation, and for sharing some of your favorite games and puzzles with our readers!

Ilona Vashchyshyn

References

Milner, S. (n.d.). *Why puzzles?* Retrieved from http://susansmathgamesca.ipage.com/why-puzzles/

Milner, S. (2013). Puzzles in my life. *CMS Notes, 45*(5).

Pacific Institute for the Mathematical Sciences. (2014, March 27). Susan Milner is awarded the PIMS 2014 Education Prize. Retrieved from http://www.pims.math.ca/news/susan-milner-awarded-pims-2014-education-prize

Quickfire Challenges to Inspire Problem Solving [1]

Suzanne R. Harper and Dana C. Cox

In our attempts to incorporate problem solving into our mathematics courses, we have found that student ambition and creativity are often hampered by feelings of risk, as many students are conditioned to value a produced solution over the actual process of building one. At times, the fear that a solution will not be found (and thus an assignment not completed) is so great that it seems to impair our students' willingness to ask and pursue ambitious mathematical questions. This fear can also impair students' ability to create models for data or natural phenomena or to move beyond the boundaries of what they already know into an area where creativity is imperative and where insight may be profound.

> "Eliminating risk is neither possible nor desired. To retreat to safety is to eliminate all struggle or conflict in the learning process."

Eliminating risk is neither possible nor desired. To retreat to safety is to eliminate all struggle or conflict in the learning process. NCTM's Principles to Actions: Ensuring Mathematical Success for All (2014) promotes productive struggle for students grappling with complex ideas or relationships, noting that productive struggle provides students with "opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions" (p. 48).

The challenge, then, is to understand and help students manage the risks associated with creative mathematical work. Problem solving in the context of an assigned project or a long-term activity can seem to portray an alternate universe of mathematical behavior, severed from daily activity. The time required to delve deeply into a problem prohibits these opportunities on a regular basis, so they retain a feeling of foreignness. We would add that when the weight of assessment is based on a completed product, the projects are likely to be tripwires for the risk cycle we have observed and described.

Breaking the problem-solving cycle into smaller tasks that focus on different strategies may have the unintended consequence of turning the act of problem solving into a series of procedures. As an alternative to fracturing a complex process into discrete parcels, we present here a structure for short-term work that maintains the coherence of the problem-solving process, places emphasis on the process rather than the produced solution, and creates an artifact of that process for further reflection and discussion.

Our structure is based on the quickfire challenge that is common and popular on competitive cooking and design shows. We will explain what we mean by "quickfire challenge," describe one quickfire challenge that we have incorporated into our instruction, and share how our classroom norms influence its use. We will conclude by brainstorming ways to use the student reasoning and products produced in a mathematical quickfire challenge to inform instruction and make decisions about future activity.

---

[1] Reprinted with permission from Quickfire Challenges to Inspire Problem Solving, *Mathematics Teacher 110*(9), copyright 2017 by the National Council of the Teachers of Mathematics (NCTM). All rights reserved.

A Mathematical Quickfire Challenge: The Kaleidoscope Task
The quickfire challenges in cooking and design competitions have three hallmarks:

1. They are conducted in a short time period (minutes, not hours).
2. They give participants a chance to use instinct and technique.
3. They result in formative (not summative) feedback on techniques or related skills (no one is removed from the competition as a result of his or her performance, although some participants may be offered rewards).

A mathematical quickfire challenge (MQC), however, has three additional hallmarks:

1. It should be framed as an open-ended problem that permits multiple strategies and solutions.
2. It should be defined loosely so that students make decisions about where to put their focus, creating an environment that fosters diversity of thinking rather than conformity.
3. It should enable the act of problem posing. By reformulating the task as a specific problem to be solved, solvers are able to imprint their own perceptions of a real-life phenomenon into the task (Silver 1997).

The following Kaleidoscope task meets our criteria to be considered an MQC:

> With a partner, create one dynamic geometry sketch that looks like a kaleidoscope image. You will have 15 minutes to create your sketch before demonstrating it to the group.

First, the task meets all three requirements of a general quickfire challenge: It is fast, focuses on instinct and technique, and results in formative feedback. Here is why. The task is enacted in just 15 minutes of class time, with little time spent launching the task. Students who do not have a working knowledge of a kaleidoscope can rely on their partners' expertise or can engage in a digital search for information. Students in our class use interactive geometry software (IGS) regularly and with flexibility. This task enables them to use familiar techniques of construction to create, transform, and animate geometric objects. Last, students are not asked to submit a product; rather, they present results to the whole group for feedback and improvement. We use the following prompts to focus those presentations on the process of problem solving rather than on the end product:

> "As facilitators of the discussion, we work to help classmates offer feedback, which often generates powerful, collaborative problem-posing and problem-solving sessions."

(a) How does your sketch resemble a kaleidoscope pattern?
(b) What mathematical or technological problems emerged as you worked?
(c) Did you arrive at any mathematical or technological insights while working?
(d) Did any limitations or struggles prevent you from realizing your vision?

As facilitators of the discussion, we work to help classmates offer feedback, which often generates powerful, collaborative problem-posing and problem-solving sessions. Presenters commonly spur their classmates into helping them improve their original designs.

The task also meets our criteria for a mathematical quickfire challenge. The task does not focus on a specific answer and has no expected strategy. Nonroutine, novel, or loosely defined problems hold great potential for inspiring ambitious problem solving. Here, there is no idealized product or exemplar for which all are aiming, and there are multiple productive paths to take. These paths may include brief Internet searches for stimulus, but the short time frame and the emphasis on process rather than product decrease the motivation to simply cut-and-paste someone else's solution. The task is loosely defined, because the direction to take depends on the facets of a kaleidoscope pattern that students wish to represent. Given the short time frame, capturing the absolute and true essence of a kaleidoscope image would be impossible. There is room for decision making, and there is potential for diverse thinking.

> "Nonroutine, novel, or loosely defined problems hold great potential for inspiring ambitious problem solving. Here, there is no idealized product or exemplar for which all are aiming, and there are multiple productive paths to take."

Problem posing enters the process as students pose questions of both mathematical and technological possibility. These often sound like, "I wonder if we can get it to . . ." or "How might we use rotation here instead of reflection?" In deciding which tools to use in the construction process, mathematics is essential. Mathematical actions that double as technological actions include making a polygon or circle; transforming shape through reflection, rotation, and dilation; and animating subsets of points or figures. These actions, taken within the software, are inseparable from mathematical reasoning.

The mathematical content that is central to the Kaleidoscope task is found in high school standards related to congruence and transformations. Ample opportunity exists for students to explore the mathematics of transformation as well as the relationships of transformed shapes to their original images. The task is technology dependent, although this is not required of all MQCs. (Later we will describe additional ideas for MQCs that are outside the domain of geometry and that are not technology dependent.)

Pedagogical Components: Technology and Classroom Norms
The task was enacted in a technology-rich classroom with students who were capable of using interactive geometry software. The IGS environment is key in this task, because the creation of geometric objects and access to geometric tools enables students to realize a sketch more quickly and reliably than other methods (Graf and Hodgson 1990). More so, the ability to animate geometric objects enables a fuller range of sketches to be created, including some that capture the dynamic and randomly shifting qualities of kaleidoscopic images.

Our classroom norms of collaboration, communication, and student mathematical authority support our use of the Kaleidoscope task. Students often work in small groups and use IGS as both a communication and collaboration tool. Students in our classes communicate mathematically with us and with one another as they seek out feedback throughout their learning process. IGS provides feedback to the students and encourages them to design sketches iteratively, tweaking and adjusting features on the basis of comparisons between their intended result and the results given by their sketch (Laborde et al. 2006). This supports our students in establishing and retaining mathematical authority during instruction.

Debreifing

Here, we will provide a condensed version of our classroom discussion, using kaleidoscope sketches created by students. We highlight the power of each of the four questions from the list above.

How does your sketch resemble a kaleidoscope pattern?

The sketches were diverse, and each group identified different facets of a kaleidoscopic pattern to represent. Many sketches were static images (see Figure 1). Bo and Luke's sketch was based on their desire to create an image with reflectional symmetry. They began with a square and segments marking lines of symmetry. Those line segments were used as lines of reflection as they created and arranged other polygons. Circles were used "to add depth to the picture," and Bo further described their process in this way:

> We plotted a bunch of different points that had to do with intersections, bisectors, etc. . . . The point is, what we did on one side, or one-fourth of the circle, is symmetric throughout the circle.

Their stated intentions and comparison to their final product is a powerful formative assessment: hearing them, we gained insights into Bo and Luke's conceptions of symmetry and how they linked to the technological reflections the boys constructed. Their comments also gave us a chance

Figure 1: Bo and Luke used reflections to create a symmetric (and static) kaleidoscope pattern

to point out places in their design that could be improved to realize that symmetrical vision. In our experience, other students will take on this work in class discussion, engaging in the act of challenging the mathematical claims and arguments of their classmates (CCSS. Math.Practice.MP3, CCSSI, 2010).
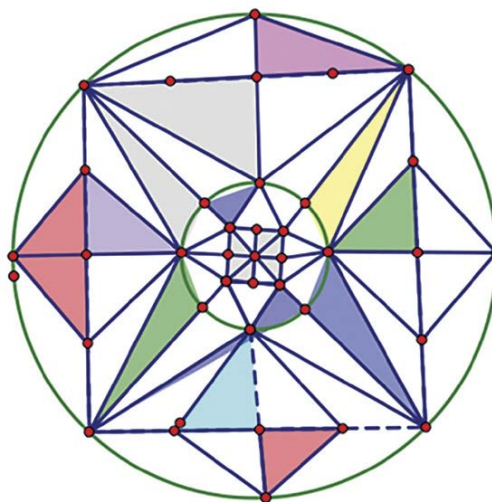

Figure 2: Desmond and Lucy created a boundary-defying kaleidoscopic image

Other sketches, such as those in Figures 2 and 3, were static as well as dynamic and focused on representing other facets, such as random motion, rotational motion, dilation, and reflection. As Bo and Luke compared their sketch to others that were presented, they chimed in that animation was something they had wanted to try, too, but they had found it difficult to control the motion and make it behave in expected ways. Furthermore, they regretted to some degree using reflections as the basis for their sketch and not finding ways to incorporate rotational symmetry.

What mathematical or technological problems emerged as you worked?

Lucy was extremely satisfied with the way one static image looked and also with the way that

she and Desmond were able to animate the shape, causing the shape to shift and move with rotational symmetry as well as dilation (see Figure 2). However, in spite of being satisfied with her perceived success with the task, Lucy described a conflict between their sketch and her conceptions of the real-life version. She focused on the jagged edges at the circumference of the outer circle:

> When the kaleidoscope goes, at times, it'll go outside of the circle. Which, with a real kaleidoscope this isn't possible because you can only see within the circle in the kaleidoscope that you are moving.

This observation pointed out a technological problem for Lucy and Desmond (and later, for the class), which can be stated in this way: How can we give the illusion that the image stops beyond the boundaries of the largest circle? However, it can also be stated as this: How can we prevent the images from moving beyond the boundaries of the largest circle? Either problem has the potential to engage the class in collaborative problem solving. Having shared all of the files previously, classmates could immediately jump into the sketch, develop and test conjectures, and move the problem beyond the initial mathematical quickfire challenge.

Did you arrive at any mathematical or technological insights while working?
Olivia and Abby created an animated sketch that had both reflectional and rotational symmetry (see Figure 3). Abby observed,

> Olivia and I both remember that when we were younger we would have little kaleidoscopes made out of paper towel tubes. When you would rotate them you'd see new shapes. We thought that was so cool. We wanted to make sure that [aspect of a kaleidoscope] was included [in our sketch].

As they viewed their animated sketch, Abby expressed a new perspective on the sketch when she noted, "When you look in a kaleidoscope, what you do is turn it." Abby took over the mouse and targeted a specific point on the original circle. When she dragged the point, the circle changed size and the objects within the circle moved, creating an illusion of rotation about the center of the circle. This visual effect prompted Abby to think about the sketch from a new perspective and to convey a new problem: How can we "turn" the sketched kaleidoscopic image?

Olivia was unable to rectify images within the animated sketch with Abby's statement, "What you do is turn it." Hence, Abby suggested they do a Google search to find a better way to convey this insight to Olivia. As Olivia viewed search results produced on the screen, she noted, "I see what

Figure 3: Olivia and Abby made a turning kaleidoscope

you mean; you rotate the entire thing." At this instant, the students implicitly posed a problem to be solved, namely, What technological processes are desired and available to "turn" a sketched kaleidoscopic image?
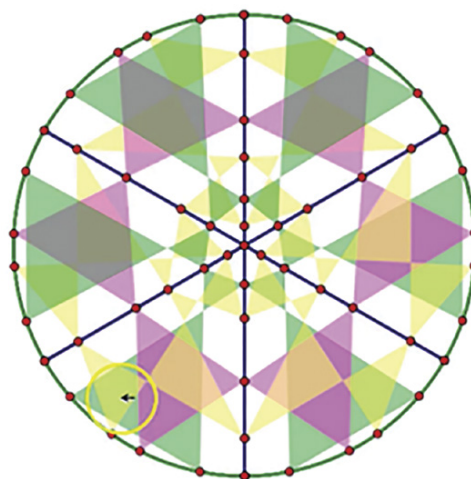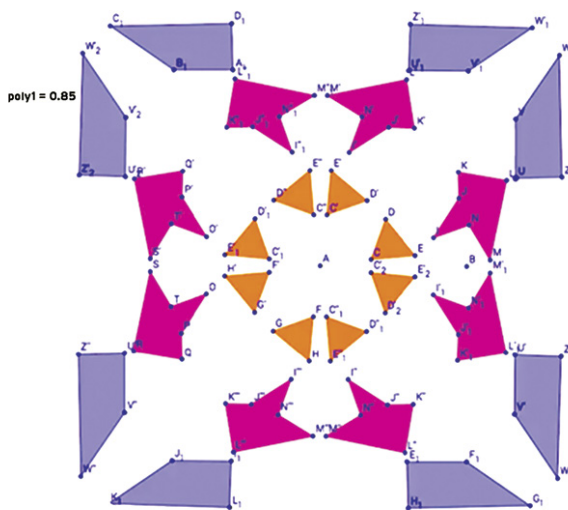
Figure 4: Amy and Bernadette began with three differently colored polygons

Did any limitations or struggles prevent you from realizing your vision?
Amy and Bernadette focused on the lack of motion or animation in their sketch as a limitation. They described their construction process as creating three lines of reflection: "We made an x-axis and y-axis and a line 45 degrees splitting the two" (see Figure 4). They then created three initial polygons (each shaded in a different color) and "reflected them over all of the lines." While this created a static image, their ambition urged them to go for a more dynamic version, but they stated that "we couldn't figure out how to animate." The two students dragged a single vertex from one polygon, and all the reflected images moved while the remaining polygons remained static. The students could not seem to manage to move all three polygons simultaneously:

> We tried to make sliders to make them move on their own, but we had a lot of trouble figuring that out. If we were able to do that, we would be able to make it look more like a kaleidoscope.

In this sense, their technological knowledge limited the pair in their pursuit of an ambitious sketch.

Moving On
Knowing this was not the finished product, student creators were open to receiving feedback to refine their sketches in subsequent iterations. "Communicating about one's thinking during a task makes it possible for others to help that person make progress on the task" (NCTM 2014, p. 49). Each presentation left the class with questions. These questions are new beginnings for students, and the task at hand can be to make improvements.

There are many ways to proceed with instruction, depending on your goals. If your goal is to assess an individual's awareness of the problem-solving process or mathematics within a sketch, you might choose to have each student create a video reflection in which they talk about how they were inspired to improve their sketch or they describe specific insights they reached that day. Another route we find compelling would be to invite individuals to improve on a sketch, be it their own or a classmate's, and share their revision as a digital file. This gives students the opportunity to take a more long-term design approach. After they have developed an ambitious sense of the sketches that can be created, they can list open questions or problems. It also continues to foster diverse thinking, as significant variations in the improvements may have been discussed.

This article has focused on a specific MQC as an illustration of a general design and method, but we do not wish to limit the imagination. Asking students to build or draw two- and threedimensional shapes that fit given criteria is another topic within geometry that might

lend itself to an MQC. However, this is not a domain-specific assignment; algebra, calculus, and statistics certainly have topics that would lend themselves to development. For example, modeling tasks are particularly well-suited to the design and implementation of quickfire challenges. Providing students with compelling data and asking them to create mathematical models and predictions would be a very appropriate context for MQC development. This could be brought into an algebra course if the data supported conversations about families of functions that might be used in those models. As you incorporate these challenges into your instruction, we would love to hear about what you have developed.

References

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics.* Washington, DC: National Governors Association Center for Best Practices and Council of Chief State School Officers. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Graf, K.-D., & Hodgson, B. R. (1990). Popularizing geometrical concepts: The case of the kaleidoscope. *For the Learning of Mathematics, 10*(3), 42–50.

Laborde, C., Kynigos, C., Hollebrands, K., & Strässer, R. (2006). Teaching and learning geometry with technology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 275–304). Rotterdam: Sense Publishers.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

Silver, E. A. (1994). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM, 29*(3): 75–80.

❖

Suzanne R. Harper, harpersr@miamioh.edu, and Dana C. Cox, dana.cox@miamioh.edu, teach prospective mathematics teachers at Miami University in Oxford, Ohio. Their professional research interests include the use of technology in the teaching and learning of mathematics, specifically using dynamic geometry software

How Can You Support Your Child in Mathematics? First, Value Math[1]

Jennifer Brokofsky

My career has taken me from Classroom Teacher, to Instructional Consultant, to Vice Principal, to my current position as Coordinator of Mathematics. Throughout this journey, I have had many opportunities to talk with parents about learning. By far the most common question that I get asked during these conversations is, "How can I help my child succeed in mathematics?"

There are so many possible answers to this question. I am going to share with you my efforts, as a parent, to create the conditions for my children to succeed in mathematics. These possibilities are not intended to be a checklist of everything you *should* do, but rather suggestions of what you *could* do.

> "As with most things, your child takes their cues from you. Their beliefs, values, and attitudes often reflect *your* beliefs, values, and attitudes."

I am going to start with what I believe the most important thing you can do… *value math*. As with most things, your child takes their cues from you. Their beliefs, values, and attitudes often reflect *your* beliefs, values, and attitudes. If you think something is important, very likely, so will they. As a teacher, I would inwardly groan every time I heard a parent say "It's okay that my child is not good at math… I wasn't." This message instantly devalues mathematics and lets your child know that they can devalue it, too. By doing this, parents create a constant struggle between the teacher and their child every time they engage in learning mathematics. As such, my number one recommendation would be to not create this battle. Instead, let your child know that you value math.

With my own kids, I constantly reinforce 5 key messages about valuing mathematics.

1. **Math is important**. I think math is important in the world, and I let my kids know it every chance I get. When I see math, I point it out. I talk to my kids about future career opportunities for them and what mathematics skills they would require to make that happen. I share with them how I use math to manage our household, where I use it in my career, and where it exists in the careers of others. My children know that math opens opportunities for them, and that they need it to be engaged and informed citizens.

2. **Math is fun**. I am constantly looking for ways to "play" with mathematics. I want my children to want to engage in mathematics and, as with most things, if it is fun, they will want to do it. I look for mathematics games, puzzles, and challenges that they can do by themselves, or that we can do together. Our closet is full of games

---

[1] A prior version of this article was published on March 24, 2015 on Jennifer's blog, *Learning Out Loud* (https://jenniferbrokofsky.wordpress.com/2015/03/24/how-can-you-support-your-child-with-mathematics-first-off-valuemath/). Reprinted with permission.

that build mathematical understanding and that are fun. (See my article "Building on Mathematical Thinking Through Play!" in Volume 1, Issue 4 of *The Variable* for some recommendations.)

3. **Math is hard (sometimes)**. My kids need to understand that sometimes math is challenging, and in these cases, persistence is necessary. I expect my children to face these challenges with a positive attitude and determination. I am always there to help them when needed, but my job is not to rescue them. When my kids hit a question that is challenging, I let them struggle. Yep, I said it… *I let them struggle*. Instead of swooping in with the answer, I choose to stand back and just offer encouragement: "You can do this!" " I believe in you!" "Do the best you can!" "I am so proud of you for not quitting!" "Look back to see if you can find a similar problem. How do you think you should do it?" This standing back is sometimes hard to do, but trust me… the empowerment they feel when they figure it out on their own is so worth it. They learn the math, but more importantly, they learn that they are capable of meeting challenges and that persistence is necessary. Of course, this does not mean that I never help them. Sometimes, after what I feel is an appropriate amount of struggle, I do step in. My goal is not to *never* help them. Rather, my goal is to build their persistence and their confidence in themselves first.

> "Instead of just swooping in with the answer, I choose to stand back and offer encouragement: 'You can do this!' 'I believe in you!' 'Do the best you can!'"

4. **Math is problem solving**. Math is more than just computation. Math is about encountering a problem and then using mathematical thinking to figure it out. I often place problems in front of my kids and ask them for their advice. These aren't problems that are written on paper about two trains leaving a station… or other things you may see in textbooks. Instead, these are problems that I encounter in life. For example, the other day I needed to drive my 7-year-old daughter to her hockey practice, and I had to figure out when to leave the house. The easy thing would have been to just figure it out on my own, but instead, I saw this as an opportunity to problem solve with my kids. I called my 7-year-old and 11-year-old over and asked them what time we should leave. Because this wasn't our first ever attempt at problem solving, they very quickly identified 3 pieces of information they needed… how long would it take to drive to the rink, how early we wanted to be there, and what time the practice began. Once they had the information they needed, they worked together to find a solution. The best part was that we could actually use their solution and get authentic feedback on whether their solution worked. We arrived on time, but more importantly, my kids learned that math is alive and relevant!

> "The easy thing would have been to just figure out the answer on my own, but instead, I saw this as an opportunity to problem solve with my kids.'"

5. **Math is reasoning**. Math is about thinking logically and making sense of situations. By looking for opportunities that allow my kids to engage in mathematical experiences in and out of our home, I am trying to enhance their ability to reason. My kids know that math is more than finding a number to an equation… it is about thinking. When I look for mathematical opportunities, I am

really looking for opportunities for them to think… with me, with others, and most importantly for themselves. One example is providing them with the choice of deciding what to wear for the day. My kids have learned to choose their clothing based on the weather conditions of the day. I share with them the forecast each day, along with the current conditions. Then I ask the question, "What would be the best choice in clothing for today?" When they were little, I would often use words like cold, hot, or warm to describe the temperature. I also talked about the appropriate clothing for cold, hot, and warm weather. Now, all I do is give them the forecast and let them make their choices. With my seven-year-old, I still sometimes veto the choices, but for the most part she is expertly reasoning her way through the possibilities with the information she has been given.

Valuing math is so very important to helping your child succeed. I really do believe it is the most important among all of the suggestions I can give parents. My other two suggestions would be to *talk* math and to *play* math with your child. (See my article "Building on Mathematical Thinking Through Play!" in Volume 1, Issue 4 of *The Variable*, where I share my favorite games for supporting mathematics learning.)

◈

Jennifer Brokofsky is the K-12 Coordinator of Mathematics for Saskatoon Public Schools. She is passionate about mathematics education, and believes in empowering students and teachers to feel ownership of, and become deeply engaged in their own learning. Her Masters work in Educational Technology and Design strongly influences her practice and her belief in the importance of technology as a tool to enhance and extend learning opportunities for all.

Intersections
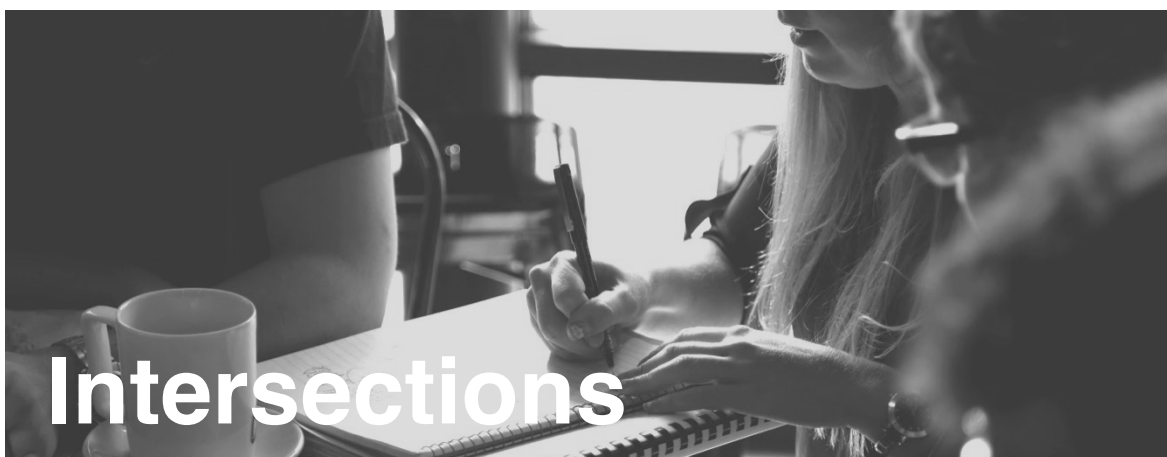
In this column, you'll find information about upcoming math education-related workshops, conferences, and other events. Some events fill up fast, so don't delay signing up!

For more information about a particular event or to register, follow the link provided below the description. If you know about an event that should be on our list, please contact us at thevariable@smts.ca.

Within Saskatchewan

Workshops

Using Tasks in High School Mathematics
November 20, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Using tasks in a high school mathematics classroom can provide rich opportunities for differentiated learning and authentic assessment. How do we choose tasks that meet both curricular outcomes and student needs? Tasks allow students to enter mathematics where they are at and extend their learning. In this workshop we will look at a variety of resources for finding good high school tasks. We will also reflect and discuss what planning and teaching moves can assist in maximizing student learning through mathematics tasks.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/using-tasks-high-school-mathematics-0

Structures for Differentiating Middle Years Mathematics Part 1 and 2
November 28, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

We know that assessing where students are at in mathematics is essential, but what do we do when we know what they don't know? What do we do when they DO know? Understanding does not change unless there is an instructional response to what we know from that assessment. The question we ask ourselves is how we might respond to individual needs without having to create completely individualized mathematics

programs in our classrooms. This workshop will include significant planning time for teams to get started in their own differentiation plans. It is also suited for educators who have already attended the Structures for Differentiation workshop.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/structures-differentiating-middle-years

Using Tasks in Middle Years Mathematics
January 15, 2018, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Using tasks in a middle years mathematics classroom can provide rich opportunities for differentiated learning and authentic assessment. How do we choose tasks that meet both curricular outcomes and student needs? Tasks allow students to enter mathematics where they are at and extend their learning. In this workshop we will look at a variety of resources for finding good middle years tasks. We will also reflect and discuss what planning and teaching moves can assist in maximizing student learning through mathematics tasks.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/using-tasks-middle-years-mathematics-1

Beyond the Worksheet: Building Mathematical Fluency and Automaticity With Games
January 16, 2018, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

Looking for a way to increase engagement while targeting mathematical fluency in your math class? Games are a wonderful way to support fluency development, conceptual understanding, reasoning, communication skills, vocabulary and more! Come join us in exploring how to leverage high quality math games that will engage students while supporting your curricular goals.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/beyond-worksheet-building-mathematical

Early Learning with Block Play—Numeracy, Literacy, and So Much More!
January 26, 2018, Saskatoon, SK
Presented by the Saskatchewan Professional Development Unit

This is a one-day workshop for early learning educators from prekindergarten, kindergarten and Grade 1 to work collaboratively to discover and deepen their understandings around the many foundational skills that children develop during block play. Through concrete, hands-on activities, participants will experience and examine the many connections between block play and curricular outcomes, and the current research on the topic. Participants will have opportunity for reflection on their current practice, planning for block play and for creating a network of support.

See https://www.stf.sk.ca/professional-resources/professional-growth/events-calendar/early-learning-block-play-%E2%80%93-numeracy-0

Beyond Saskatchewan

NCTM Annual Meeting and Exposition
April 25-28, 2018, Washington, DC
Presented by the National Council of Teachers of Mathematics

Join more than 9,000 of your mathematics education peers at the premier math education event of the year! NCTM's Annual Meeting & Exposition is a great opportunity to expand both your local and national networks and can help you find the information you need to help prepare your pre-K–Grade 12 students for college and career success. Classroom teachers, administrators, math coaches, supervisors, college professors, and preservice teachers can all benefit from the sessions and learning at this event. Improve your knowledge and skills with high-quality professional development and hands on activities; gain insights by connecting and sharing with like-minded educators; collect free activities that will keep students engaged and excited to learn; and learn from industry leaders and test the latest educational resources.

See http://www.nctm.org/Conferences-and-Professional-Development/Annual-Meeting-and-Exposition/

Online Workshops

Education Week Math Webinars
Presented by Education Week

Once a month, Education Weekly has a webinar focusing on math. They also host their previous webinars on this site. Previous webinars include Formative Assessment, Dynamic vs. Static Assessment, Productive Struggling and Differentiation.

Past webinars: http://www.edweek.org/ew/webinars/math-webinars.html
Upcoming webinars:
http://www.edweek.org/ew/marketplace/webinars/webinars.html

Did you know that the Saskatchewan Mathematics Teachers' Society is a **National Council of Teachers of Mathematics Affiliate**? When registering for an NCTM membership, be sure to support the SMTS by noting your affiliation during registration.

Tangents
Extracurricular Opportunities for K-12 Students
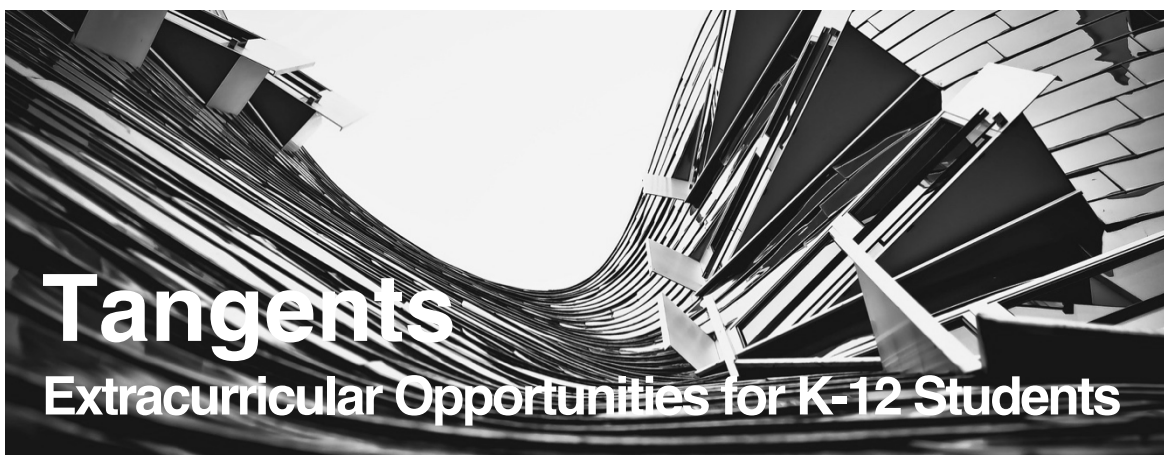
*T*his column highlights local and national extracurricular opportunities for K-12 students interested in mathematics, including collaborative, individual, online, and in-person challenges, contests, and camps. For dates, registration procedures and applications, and other information about the contests listed, please head to the contest websites, included below the descriptions. If we have missed an event that should be on our list, please contact us at *thevariable@smts.ca*.

If you are looking for contests available in other provinces, head to the Canadian Mathematical Society website (cms.math.ca/Competitions/othercanadian). The CMS also maintains a list of resources for students who are looking to build their problem-solving skills and succeed in competitions: see cms.math.ca/Competitions/problemsolving.

❖

Local Events and Competitions

Canadian Math Kangaroo Contest
Spring

The purpose of this competition is to introduce youngsters from Grade 1 to Grade 12 to math challenges in a fun and enjoyable way, thus inspiring their further interest and advancement in mathematics. The competition is held yearly in more than 40 Canadian cities. Students may choose to participate in English or in French.

Students in Saskatoon may write the contest at Walter Murray Collegiate; students in Regina may write the contest at the University of Regina. Contact Janet Christ at christj@spsd.sk.ca (Saskatoon) or Patrick Maidorn at patrick.maidorn@uregina.ca (Regina).

See https://kangaroo.math.ca/index.php?lang=en

Extreme Math Challenge
Spring

The Extreme Math Challenge is a math competition for students in Grades 7 to 10 consisting

of three rounds: individual, team, and team relay. The event will challenge students mathematically and encourage them to work together to solve math problems. Oh, and there will be prizes, too! Contact MilnerC@spsd.sk.ca for more details.

University of Regina Regional Math Camp
Spring

The Math Camp is a full-day event for students in Grades 1 through 12 who are interested in exploring the infinite frontier of mathematics beyond the school curriculum. Participants are guided by professors and students through a fun and enriching day. In a variety of grade appropriate sessions, students will explore mathematical topics in hands-on activities, games, puzzles, and more.

See https://www.uregina.ca/science/mathstat/community-outreach/mathcamp/

National Competitions

Canadian Computing Competition
Written in February
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Canadian Computing Competition (CCC) is a fun challenge for secondary school students with an interest in programming. It is an opportunity for students to test their ability in designing, understanding and implementing algorithms. Students may compete in one of two levels: Junior Level – any student with elementary programming skills; Senior Level – any student with intermediate to advanced programming skills.

See http://www.cemc.uwaterloo.ca/contests/computing.html

Canadian Team Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Canadian Team Mathematics Contest (CTMC) is a fun and challenging competition for teams of 6 secondary school students in any combination of grades. One teacher and groups of six students participate at their own school over 3 consecutive hours.

See http://www.cemc.uwaterloo.ca/contests/ctmc.html

Caribou Mathematics Competition
Held six times throughout the school year

The Caribou Mathematics Competition is a worldwide online contest that is held six times throughout the school year. Each of these days, five contests are offered, one for each of the grade levels 3/4, 5/6, 7/8, 9/10 and 11/12 and each one in English, French and Persian. Available in English, French, and Persian.

See https://cariboutests.com/

Euclid Mathematics Contest
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Euclid Mathematics Contest is an opportunity for students in their final year of secondary school and motivated students in lower grades to have fun and to develop their mathematical problem solving ability. Most of the problems are based on the mathematical curriculum up to and including the final year of secondary school. Written in April.

See http://www.cemc.uwaterloo.ca/contests/euclid.html

Fryer, Galois, and Hypatia Mathematics Contests
Written in April
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Fryer, Galois and Hypatia Math Contests are an opportunity for students to write a full-solution contest. They are fun way to develop mathematical problem solving skills through a written mathematical activity. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia).

See http://www.cemc.uwaterloo.ca/contests/fgh.html

Gauss Mathematics Contests
Written in May
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Gauss Contests are an opportunity for students to have fun and to develop their mathematical problem solving ability. For all students in Grades 7 and 8 and interested students from lower grades.

See http://www.cemc.uwaterloo.ca/contests/gauss.html

Opti-Math
Written in February
Presented by the Groupe des responsables en mathématique au secondaire

A French-language mathematics challenge for secondary students who would like to exercise and develop their problem-solving skills.

See http://www.optimath.ca/index.html

Pascal, Cayley, and Fermat Contests
Written in February
Presented by the Centre for Education in Mathematics and Computing (University of Waterloo)

The Pascal, Cayley and Fermat Contests are an opportunity for students to have fun and to develop their mathematical problem solving ability. For students in Grades 9 (Fryer), 10 (Galois) and 11 (Hypatia).

See http://www.cemc.uwaterloo.ca/contests/pcf.html

Sun Life Financial Canadian Open Mathematics Challenge
Written in November
Presented by the Canadian Mathematical Society

A national mathematics competition open to any student with an interest in and grasp of high school math. The purpose of the COMC is to encourage students to explore, discover, and learn more about mathematics and problem solving. The competition serves to provide teachers with a unique student enrichment activity during the fall term. Available in English and French. Written in November.

Approximately the top 50 students from the COMC will be invited to write the Canadian Mathematical Olympiad (CMO). Students who excel in the CMO will have the opportunity to be selected as part of Math Team Canada -- a small team of students who travel to compete in the International Mathematical Olympiad (IMO).

See https://cms.math.ca/COMC

The Virtual Mathematical Marathon
Supported by the Canadian National Science and Engineering Research Council

The virtual Mathematical Marathon has been developed by an international team of mathematicians, mathematics educators and computer science specialists with the help of the Canadian National Science and Engineering Research Council and its PromoScience program.

The main activity is a competition allowing students to enjoy solving challenging mathematical problems all year around. Students can join the game at any time and at no cost, simply creating an individual profile with an individual username and a password. Available in French.

See http://www8.umoncton.ca/umcm-mmv/index.php

Math Ed Matters
by MatthewMaddux
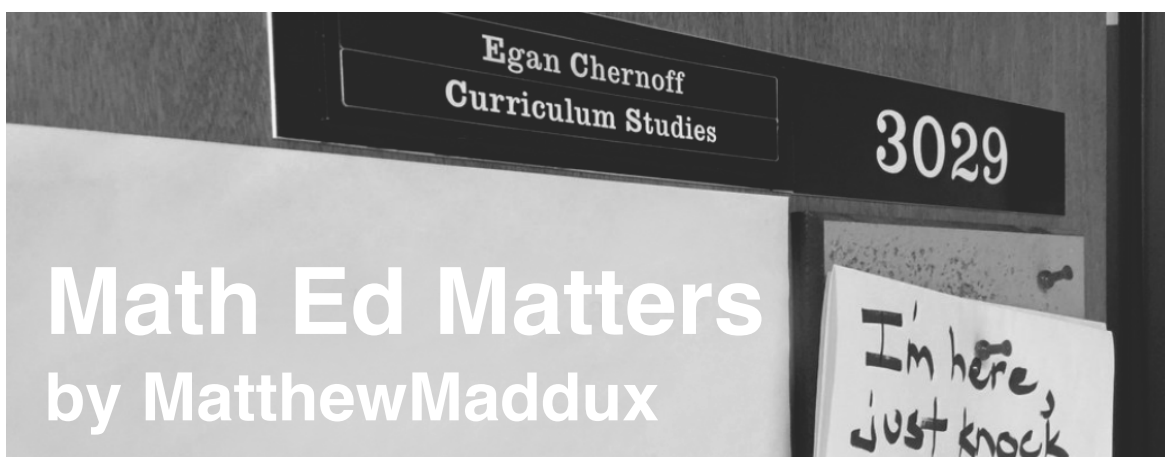
Math Ed Matters by MatthewMaddux *is a bimonthly column telling slightly bent, untold, true stories of mathematics teaching and learning.*

The Inside Joke on Math Lessons

Egan J Chernoff

I love stand-up comedy. Love it. Besides watching specials, I also listen to interviews with comedians as often as I can: Whether they're promoting their latest work or just talking with another fellow comedian on a podcast, I'm listening. Why? Well, partly for a good laugh, but also because I'm interested in how different comedians approach their craft. Louis C.K. is one comedian, writer, actor, and producer (the list goes on and on) who is often conducting interviews, thanks to the diversity of his work. In the interest of full disclosure, I'm a fan.

During his interview on *Talking Funny* (with Ricky Gervais, Chris Rock, and Jerry Seinfeld), I found out that, like his idol, George Carlin, Louis C.K. starts every year anew, with his best joke from the previous year acting as his starting point for the next. In other words, Louis C.K. retires all of his previous material every year. Even Jerry Seinfeld, star of the 1998 special *I'm Telling You for the Last Time,* appeared taken aback by this revelation. And it was this particular comment from Louis C.K. that got me thinking about the parallels between stand-up comedy and the teaching and learning of mathematics.

> "Although comedians and teachers work on different time scales, I see many similarities between a comedian's set and a math teacher's lesson. No, seriously."

At this point, I want to make it clear that, although I will be drawing a parallel between stand-up comedy and teaching mathematics, I am not referring to "edutainment," which, as the blended name suggests, is an amalgam of entertainment and education. Instead, the parallel that I wish to draw is based on the change, over time, in the material that (1) a stand-up comedian delivers to their audience and that (2) a math teacher 'delivers' to their students. Although comedians and teachers work on very different time scales, I see many similarities.

Based on what I have gathered from watching countless interviews with comedians, I will first try to paint a thumbnail sketch of how a stand-up comedian ends up with a set, for the

purpose of comparison. The stand-up comedian must, of course, begin with some material. No material, no set. However, there are often vast differences between the initial material and the material that ends up in a one-hour special that gets aired on TV or on Netflix. The initial material gets honed and honed as the comedian performs set after set after set. Based on the response received from the audience, or lack thereof, material is added, removed, or tweaked until the comedian deems it fit for primetime. This process, that is, the continuous honing of the material, is key in getting to the point where, no matter the venue, the city, or the audience, they are able to kill (to use their terminology).

As I suggested earlier, having taught mathematics for a number of years, I see many similarities between a stand-up comedian's set and a math teacher's lesson. No, seriously.

Like the stand-up comedian, a math teacher needs some material before they go in front of their 'audience' (the students in their classroom). This material, of course, is their lesson plan. However, as all math teachers and comedians know, and as future math teachers will find out, your material is never delivered exactly as it's drawn it up. Maybe one part of the lesson plan doesn't go over very well, and more time is needed in order to make sure that everyone is on the same page before you move on. Maybe a part of your lesson plan particularly resonated with your students and, recognizing this, your lesson delves a little deeper into Pythagorean triples before you move on to the other examples in your lesson plan. Maybe you realize that you will only need one example instead of the initial three that you had planned for. Or, maybe something that you thought would take ten minutes ends up taking 20. What I'm trying to say is that the implementation of a lesson plan (i.e., your lesson) rarely goes as planned, thanks to your audience. And, to be clear, this is a good thing. While the easiest way to implement a lesson plan is to deliver it exactly as planned, this also means ignoring your audience completely, and functioning as little more than a lesson-plan-delivery system. However, the influence of the audience should not only impact teaching "in the moment," but also after the lesson (set, for the comedian) has been implemented. (And, to be clear, I'm not advocating for the tail wagging the dog.)

> "While the easiest way to implement a lesson plan is to deliver it exactly as planned, this also means functioning as little more than a lesson-plan-delivery system."

Reflecting on your set (if you're a comedian) or your lesson (if you're a math teacher) after the fact is crucial to honing your craft. I've heard stories of stand-up comedians who, after completing a set, hop into their car and drive to the other side of town to deliver it again the very same evening, so that they can immediately implement the changes based on the audience's reactions. A similar thing often happens for math teachers. This is especially true for secondary math teachers, who often teach more than one section for a particular grade. For example, I once taught seven courses comprised of three classes of math 11, two classes of math 10, one class of math 12, and one class of math 12 honours. My three classes of math 11 fell on the same day, back-to-back-to-back. This set-up was akin to the comedian driving from one set immediately to another, and then immediately to another.

In between back-to-back lessons, math teachers will either actively work on some changes to their lesson plans or leave things alone. And if you don't take the time to hone your lesson plan in between classes, one thing typically happens: you enter a time warp. Because you haven't made any changes to your lesson plan, although it should (on paper) take the same amount of time to implement, the lesson tends to get shorter and shorter each time. The key reason for this is that we, as math teachers, tend to get too familiar with our

material, and don't recognize that we have a new audience with each new class. This familiarity with the material, perhaps akin to Steven Pinker's notion of the "curse of knowledge," leads to speed lessons. I remember instances in my own experience where, having implemented the same lesson plan a number of times, a lesson would take almost 20 minutes less to deliver to students by the end of the day. It wasn't until much later that I realized that my third instance of the same lesson would be much worse than the first.

Let's say, though, that you don't teach high school mathematics and, instead, teach elementary or middle-school mathematics. Because of this, you probably won't have the blessing (or the curse, depending on how you look at it) of teaching back-to-back lessons. In fact, the time frame associated with honing your craft sits at around the 12-month mark… quite enough time to completely forget about any changes that you wanted to make to your lesson!

One of the more effective ways to hone your craft on a yearly basis is to keep track of what went well and what didn't go well during your lesson. And, although teachers are notoriously pressed for time, keeping track of how your math lesson went doesn't need to be an arduous task. In fact, one of the easiest ways to do this, I found, is to use the school calendar that is (still, hopefully) given out to all students and teachers at the beginning of the school year. Comments don't need to be extensive; rather, they should get right to the point. Unfortunately, I wouldn't learn this strategy (simple, in hindsight) until later in my teaching career. (Sorry, math students who I taught early in my career!)

> "One of the effective ways to hone your craft is to keep track of what went well and what didn't go well during your lesson. Unfortunately, I wouldn't learn this until later in my teaching career."

The idea of keeping tabs on how things went during a math lesson is something that I stumbled upon after one lesson, as time marched on, wasn't getting better. Actually, it was getting worse. I remember teaching a particular lesson that I used with my two Grade 11 classes where, while things went quite well for the majority of the lesson, things derailed pretty quickly at a particular point. However, with only a few minutes between my first and second classes, after finishing my first class of the day, I turned right around and tried to teach the same lesson plan again. Sure enough, the same thing happened. The lesson was, once again, derailed, at the same point that it had derailed in the previous class (not temporally, however, thanks to the time warp that resulted from teaching the same lesson twice). Both times, I noticed that things started to slip when we had finished solving a particular example that I had chosen. I wasn't sure exactly what had happened, but I left that day knowing, without a doubt, that this example was ruining a specific portion of my lesson and that it was hard to get the 'audience' back after that point. So, as I put the lesson plan away into my filing cabinet, I made a mental note to make sure to remember to take out this particular example and to find another for next year, so that I wouldn't have the same issue arise again.

Perhaps you're getting ahead of me at this point.

Moving ahead a year in my teaching career, I considered myself lucky to be teaching three sections of math 11, as this meant a dramatic cut to my planning time. I could plan for just one course, and this would cover three of the classes that I taught. In other words, I could just teach the same lesson, three times! They gave me an inch, and I took a mile.

Smash cut to me implementing the exact same lesson plan from the year before. Smash cut to the exact same example from the year before. Now, it's not that I didn't realize that I was doing the same example from the year before, the one that didn't work out all so well and that I told myself that I needed to replace… rather, it was *when* I realized that I was doing the same example. Nope, not after the first lesson. Not after the second lesson, either. That's right: Only after the third time I gave the lesson, after the school day was over, did I realize that the example that had derailed two of my lessons the year before had derailed three of my lessons this year, simply because I had forgotten to implement the change that I told myself to make before I ever did the lesson again. Next year, I told myself, this were going to be different.

You must be getting ahead of me again.

Well, actually, things *were* different this time. I put a yellow sticky note on top of my lesson plan before I neatly filed it in my filing cabinet so that I would be able to find it easily next year. Smash cut to a year later, and there I am, opening up my filing cabinet, opening up the folder with the same lesson plan that I had now used for three years. To my surprise, there was an incoherent (before the lesson, but coherent after the lesson) yellow sticky note on the first page, which I promptly removed and filed in the recycling bin. Sure enough, this time, just before I wrote the example on the board, I remembered that this was the example that had derailed the lesson the year before, and the year before that, too. While I was able to recognize that I shouldn't do this particular example with the students, I didn't have another one to replace it. Clearly, I wasn't working on my material. Strike three.

> "Math teachers, like comedians, must be always working on their material and constantly honing their craft."

This somewhat embarrassing tale from my teaching, I hope, helps to illustrate a larger point that I am trying to make, which is that math teachers, like comedians, must be always working on their material and constantly honing their craft. In other words, as math teachers, we need to be regularly revising our lesson plans based on feedback from students, ourselves, and peers. No matter how good we think our original lesson plan (i.e., material) is for any given topic, it should only get better the more often it is implemented. This will only happen if we reflect on, and learn from the implementation of our lesson plans.

From this perspective, an original lesson plan will likely never be as good as the fifth time you implement it, under the caveat that you don't treat your lesson plan as a static entity. To borrow a phrase from the world of stand-up comedy, you need reps (i.e., repetitions of a set). Reps are important. The more reps, and the more you hone your lessons, the better you should become as a math teacher. Hopefully, after a number of years, your lesson plan, and your implementation of said lesson plan, will be good enough that you are ready, much like a stand-up comedian, for your one-hour special. To use comedy parlance, at some point, your lesson plan gets good enough that you are comfortable with your "set" for a particular topic. You might even develop a bit of a 'routine'! Of course, you won't have only one lesson plan: You'll have ten months of lesson plans, because, unlike a stand-up comedian, a teacher stands in front of an audience and does different sets every single day, multiple times a day. No easy task, but what an opportunity to hone your craft!

Some stand-up comedians take years to prepare a particular set of material for a one-hour special. The more skilled among them are able to deliver specials on a more frequent basis, but even for seasoned stand-up comedians like Louis C.K. or Bill Burr, more than one

special a year is a tall order. Future math teachers, on the other hand, will develop hundreds and hundreds of different sets (that is, lessons) that they teach each and every day, and hone each of these different lessons year after year. The comedian and the math teacher, then, are similar in that, respectively, one hones one particular set of material night after night until their set is ready for primetime, which takes about a year, while the other hones a large number of different sets, but only does so once or twice a year, for year after year. Ultimately, the math teacher and the standup comedian get to the same place, where stand-up specials equate to killer math lessons.

At this point in their path, the stand-up comedian will likely film their special in order to start selling DVDs and digital downloads so as to monetize the tremendous amount work that went into their one-hour special, which may motivate them to work on more. The path of the math teacher, of course, is slightly different. Thanks to the cyclical nature of the job, the math teacher must worry about complacency setting in once they have developed ten months of solid daily lesson plans. Fear not, though: Those new math curricula that are perpetually coming down the pipe mean that, like Louis C.K., you'll be starting everything over again soon enough.

◆

Egan J Chernoff (Twitter: @MatthewMaddux) is an Associate Professor of Mathematics Education in the College of Education at the University of Saskatchewan. Currently, Egan is the English/Mathematics editor of the Canadian Journal of Science, Mathematics and Technology Education*; an associate editor of the* Statistics Education Research Journal*; sits on the Board of Directors for* for the learning of mathematics*; serves on the International Advisory Board for* The Mathematics Enthusiast*; is an editorial board member for* Vector: Journal of the British Columbia Association of Mathematics Teachers*; and, is the former editor of* vinculum: Journal of the Saskatchewan Mathematics Teachers' Society.